Uniqueness and stability in symmetric games: Theory and Applications

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Abstract

This article develops a comparably simple approach towards uniqueness of pure-strategy equilibria in symmetric games with potentially many players by separating between multiple symmetric equilibria and asymmetric equilibria. Our separation approach is useful in applications for investigating, for example, how different parameter constellations may affect the scope for multiple symmetric or asymmetric equilibria, or how the equilibrium set of higher-dimensional symmetric games depends on the nature of the strategies. Moreover, our approach is technically appealing as it reduces the complexity of the uniqueness-problem to a two-player game, boundary conditions are less critical compared to other standard procedures, and best-replies need not be everywhere differentiable. The article documents the usefulness of the separation approach with several examples, including applications to asymmetric games and to a two-dimensional price-advertising game, and discusses the relationship between stability and multiplicity of symmetric equilibria.

Keywords: Symmetric Games, Uniqueness, Symmetric equilibrium, Stability, Industrial Organization

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1 Introduction

Whether or not there is a unique (Nash) equilibrium is an interesting and important question in many game-theoretic settings. Many applications concentrate on games with identical players, as the equilibrium outcome of an ex-ante symmetric setting frequently is of self-interest, or comparably easy to handle analytically, especially in presence of more than two players. The index theorem - often referred to as the most general approach to uniqueness in nice games (e.g. Vives (1999)) - can be hard to exploit in applications (especially with many players) as it involves analyzing the determinant of a potentially large matrix, and also requires boundary conditions to hold, which not infrequently are violated[1] by important examples.

This article presents a comparably simple approach to analyze the equilibrium set of a symmetric game in thus that the symmetry inherent in these games is exploited by separating between the two natural types of equilibria - symmetric and asymmetric equilibria. What makes this separation approach appealing for applications are its simplicity (as the problem is essentially reduced to a two-player game) and its applicability (as boundary conditions are less problematic, or best-replies may have kinks). At a more theoretical level we can also use this approach to investigate how the scope for multiple symmetric equilibria or asymmetric equilibria depends on certain parameter constellations in a game, or, more fundamentally, on the nature of the strategies of a game.

The practical usefulness of the approach for all these aspects is demonstrated with several applications. For example, we show that a symmetric game with a two-dimensional strategy space (e.g. price and quality) can never possess strictly ordered asymmetric equilibria (where one player sets both a higher price and quality) if either price or quality is non-decreasing in the opponents' actions. Further, it is shown that sum-aggregative symmetric games with homogeneous revenue functions (e.g. contests) naturally have a unique symmetric equilibrium, or that the classical assumption $c'' - P' > 0$ is key for uniqueness in the Cournot model *because* it rules out the possibility of asymmetric equilibria. The separation approach is also used to discuss uniqueness in a two-dimensional information-pricing game both at an abstract level and in a particular example.

[1]Consider the examples in sections 4.3 and 4.4.

Analyzing the equilibrium set of a symmetric game may not only be of self-interest, but matters also because we can learn more about the equilibrium set of asymmetric variations of that game. For example, it is shown that there is an intimate link between the (in)existence of asymmetric equilibria in a symmetric game and the order of the equilibrium strategies in asymmetric variations of the game. Moreover, uniqueness of equilibria in a symmetric game is preserved under small asymmetric variations of the game. Finally, the relationship between symmetric stability (i.e. stability under symmetric initial conditions) and the multiplicity of symmetric equilibria is investigated. It is shown that the existence of exactly one symmetric equilibrium is the same formal property as symmetric stability in one-dimensional nice games, and symmetric stability implies that there only is one symmetric equilibrium in higher-dimensional games. Moreover, in one-dimensional games a single symmetric equilibrium is globally stable under symmetric adjustments. To summarize, the separation approach provides us with powerful and yet comparably simple tools to examine the equilibrium set of symmetric games that may have eluded a formal assessment so far e.g. because of the sheer complexity of the problem under standard methods.

The paper is structured as follows. After introducing the notation, the separation approach is developed in section 3, and section 4 applies the approach to several examples. Finally, section 5 is concerned with the relationship between stability and the multiplicity of symmetric equilibria.

2 Basic notation and assumptions

Consider a game of $N \geq 2$ players indexed by $1, ..., N$. Let $x_g \equiv (x_{g1}, ..., x_{gk}) \in S(k)$ denote a strategy of player g, where $S \equiv S(k) = \times_{i=1}^{k} S_i$ with $S_i = \left[0, \bar{S}_i\right] \subset \mathbb{R}$. The interior of S_i is non-empty and denoted by $Int(S_i)$. All players have the same compact and convex strategy space S, and the joint strategy space is S^N. Throughout this article strategies are defined to be pure strategies. For any player g the vector $x_{-g} \in S^{N-1}$ is a strategy profile of g's opponents. The payoff of g is represented by a function $\Pi^g \left(x_1, ..., x_g, ..., x_N\right) \equiv \Pi \left(x_g, x_{-g}\right)$. Unless stated otherwise, the following properties of the payoff functions are assumed throughout this article:

- Symmetry: Payoff functions are permutation-invariant[2], meaning that for any permutation σ of $\{1, ..., N\}$, payoff functions satisfy

$$\Pi^g (x_1, ..., x_g, ..., x_N) = \Pi^{\sigma(g)} \left(x_{\sigma(1)}, ..., x_{\sigma(g)}, ..., x_{\sigma(N)} \right)$$

  on S^N, i.e. all players have the same payoff function.

- $\Pi (x_g, x_{-g}) \in C^2 \left(O^N, \mathbb{R} \right)$, where $O \supset S$ is open in $\mathbb{R}^k$, and Π is strongly quasiconcave[3] in $x_g \in S$ for any $x_{-g} \in S^{(N-1)}$.

Let $\nabla \Pi^g(x)$ denote the gradient (a k-vector) of $\Pi(x_g, x_{-g})$ with respect to x_g, and $\nabla F(x) \equiv (\nabla \Pi^g(x))_{g=1}^N$ is the pseudogradient (a Nk vector, Rosen (1965)). The triple $\left(N, S(k)^N, \Pi \right)$ denotes a symmetric, differentiable k-dimensional N-player game, and the formulation "a game" in text refers to this triple.

Player g's best reply $\varphi^g(x_{-g})$ solves $\max_{x_g \in S} \Pi(x_g, x_{-g})$. As a consequence of the above assumptions, individual best-replies $\varphi(x_{-g}) \equiv \varphi^g(x_{-g})$ as well as the joint best-reply $\phi (x_1, ..., x_N) = (\varphi (x_{-1}), ..., \varphi (x_{-N}))$ are continuous functions. A (Nash) equilibrium is a fixpoint (FP) $\phi(x^*) = x^*$, $x^* \in S^N$. If $x_1^* = ... = x_N^*$, then the equilibrium is symmetric. Note that any symmetric equilibrium $x^* \in S^N$ is identified e.g. by its first projection $x_1^* \in S$. To find symmetric equilibria a simplified approach, called Symmetric Opponents Form Approach (SOFA) hereafter, is useful (applied e.g. by Salop (1979), Dixit (1986) or Hefti (2012)). The SOFA takes an arbitrary indicative player (e.g. $g = 1$), and then restricts all opponents to play the same strategies, i.e. $\bar{x}_{-g} = (\bar{x}, ..., \bar{x})$, where $\bar{x} \in S$. Let $\tilde{\Pi} (x_1, \bar{x}) \equiv \Pi^1 (x_1, \bar{x}_{-1})$, $\tilde{\Pi} : S^2 \to \mathbb{R}$, with corresponding best-reply function $\tilde{\varphi}(\bar{x}) \equiv \varphi(\bar{x}_{-1})$. Note that $\tilde{\varphi}$ inherits continuity from φ^1. The derivative of $\tilde{\varphi}$ at $\bar{x}$ is denoted by $\partial \tilde{\varphi}(\bar{x})$. The following result, which we include mainly for clarity and completeness, is a direct consequence of the above definitions and assumptions:

Proposition 1 *x^* is a symmetric equilibrium iff $x_1^* = \tilde{\varphi}(x_1^*)$.*

[2]See Dasgupta and Maskin (1986).

[3]Strong quasiconcavity means that $z \cdot z = 1$ and $z \cdot \frac{\partial \Pi^g(x)}{\partial x_g} = 0$ imply $z \cdot \frac{\partial^2 \Pi^g(x)}{\partial x_g \partial x_g} z < 0$ (see Avriel et al. (1981)).

As an immediate consequence of the assumptions imposed on Π and the strategy space we obtain the following existence result:

Proposition 2 *A symmetric game has a symmetric equilibrium, and the set of symmetric equilibria is compact.*

Proof: Continuity and strong quasiconcavity of Π together with compactness and convexity of S imply the continuity of $\tilde{\varphi}(\bar{x})$. Existence and compactness then follow from $\tilde{\varphi} \in C(S, S)$ and the Brouwer FP theorem.

■

3 The separation approach

The standard approaches to verify uniqueness (see e.g. Vives (1999)) are i) the contraction mapping approach, ii) the univalence approach and iii) the Poincare-Hopf index theorem approach. Obviously, these methods can be applied to symmetric games. Their shortcomings are that they may be restrictive, involve boundary conditions or require calculating the determinant of potentially very large matrices. Furthermore, we cannot use these methods to investigate, for example, what parameter constellations might cause a game to have multiple symmetric equilibria versus asymmetric equilibria. Moreover, multiplicity of equilibria in symmetric games can mean multiple symmetric equilibria, asymmetric equilibria or both. This simple observation is the starting point of the now proposed approach towards uniqueness in symmetric games. By taking advantage of the dichotomy of equilibrium types and the symmetry in the game, it is possible to reduces the dimensionality of the FP problem from an N-player to a two-player game. First, we will look at the possibility of multiple symmetric equilibria, then we turn to asymmetric equilibria.

3.1 Multiple symmetric equilibria

For investigating whether or not there are multiple symmetric equilibria the index theorem, applied to the SOFA, gives a powerful tool, especially as this restricted version of the index theorem may still be applicable if the unrestricted version is not. While the adaptation of the index theorem to investigate the multiplicity of symmetric equilibria is not completely surprising to readers very familiar with index theory, the results by themselves are useful for further exploring e.g. the relationship between stability and uniqueness of symmetric equilibria (see section 5). Moreover, the index results below also indicate how to proceed in the case, where the symmetric version of the index theorem cannot be applied, e.g. because boundary conditions fail, as is the case in some applications.[4]

Let $Cr^s = \{x_1 \in S : \nabla\tilde{\Pi}(x_1) = 0\}$ denote the set of critical points, where $\nabla\tilde{\Pi}(x_1)$ is the gradient of $\tilde{\Pi}(x_1, \bar{x})$ with respect to x_1, evaluated at $\bar{x} = x_1$. Further $\nabla\tilde{\Pi}(x_1) : S \to \mathbb{R}^k$, $x_1 \mapsto \nabla\tilde{\Pi}(x_1)$ is a C^1-vector field with corresponding $k \times k$ Jacobian $\tilde{J}(x_1)$. The index $I(x_1)$ of a zero of $\nabla\tilde{\Pi}$ is defined as $I(x_1) = +1$ if $Det(-\tilde{J}(x_1)) > 0$ and $I(x_1) = -1$ if $Det(-\tilde{J}(x_1)) < 0$. I call a symmetric game an index game if i) $\nabla\tilde{\Pi}$ has only regular zeroes[5] and ii) $\nabla\tilde{\Pi}$ points inwards at the boundary of S.

Theorem 1 *There is an odd number of symmetric equilibria in an index game, and only interior symmetric equilibria exist. Moreover, there is only one symmetric equilibrium iff for $x_1 \in Cr^s$ one of the following conditions is satisfied: i) $Det(-\tilde{J}(x_1)) > 0$, ii) $Det\,(I - \partial\tilde{\varphi}(x_1)) > 0$, iii) $\prod_{i=1}^{k} (1 - \lambda_i) > 0$, where λ_i is an eigenvalue of $\partial\tilde{\varphi}(x_1)$.*

Proof: Oddness, $x_1 \in Int(S)$ and i) are index theorem results (see e.g. Vives (1999)). To see ii) decompose $\tilde{J}$ as $\tilde{J} = A + B$, where $A = \frac{\partial^2\tilde{\Pi}(x_1,\bar{x})}{\partial x_1 \partial x_1}$ and $B = \frac{\partial^2\tilde{\Pi}(x_1,\bar{x})}{\partial x_1 \partial \bar{x}}$, both evaluated at $\bar{x} = x_1$. The Implicit Function Theorem (IFT) asserts that $\partial\tilde{\varphi} = -A^{-1}B$, which shows that $Det(-\tilde{J}(x_1)) > 0 \Leftrightarrow Det\,(I - \partial\tilde{\varphi}(x_1)) > 0$. Finally, iii) is equivalent to ii) because for any eigenvalue λ of $\partial\tilde{\varphi}(x_1)$ the number $(1 - \lambda)$ is an eigenvalue of $I - \partial\tilde{\varphi}(x_1)$.

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Note that the dimensionality of the objects involved in theorem 1 is k rather than Nk. Moreover, regularity and the symmetric boundary conditions evoked in the definition of a symmetric index game are weaker than the corresponding regularity and boundary conditions of the unrestricted

[4]See section 4.3 for a one-dimensional, and section 4.4 for a two-dimensional application, where the symmetric index theorem boundary conditions naturally fail.

[5]$Det(\tilde{J}(x_1)) \neq 0$ whenever x_1 is a zero of $\nabla\tilde{\Pi}$.

vector field induced by ∇F, i.e. the index conditions may be satisfied under $\nabla\tilde{\Pi}$ even if they are violated under ∇F. For example, the conventional index theorem cannot be applied to the two-player game with FOC's $\nabla\Pi^i = -x_i - x_j$ and $S = [-1, 1]$ as there are *no* regular points. But as $\nabla\tilde{\Pi}(x_1) = -2x_1$ and $\tilde{J}(x_1) = -2$ the symmetric index theorem (theorem 1) immediately tells us that $x = 0$ is the only symmetric equilibrium, which obviously is correct. From the different conditions in theorem 1 several new conditions asserting that only one symmetric equilibrium exists can be derived.

Corollary 1 *There exists only one symmetric equilibrium if for $x_1 \in Cr^s$ one of the following local conditions is satisfied: i) $\tilde{J}(x_1)$ has a dominant negative diagonal, ii) there is a matrix norm $\|\cdot\|$ such that $\|\partial\tilde{\varphi}(x_1)\| < 1$.*

Proof: To see i) consider the decomposition $\tilde{J} = \tilde{A} + \tilde{B}$, where $\tilde{A}$ is a diagonal matrix with $\frac{\partial\tilde{\Pi}_i(x_1,x_1)}{\partial x_{1i}}$ as its ii-th entry. Hence $Det(-\tilde{J}(x_1)) > 0 \Leftrightarrow Det\left(I + \tilde{A}^{-1}\tilde{B}\right) > 0 \Leftrightarrow \prod_{i=1}^{k}(1 + \tilde{\lambda}_i) > 0$, where $\tilde{\lambda}$ is eigenvalue of $\tilde{A}^{-1}\tilde{B}$. But diagonal dominance of $\tilde{J}$ implies that every row sum of the absolute values of the entries of $\tilde{A}^{-1}\tilde{B}$ must be strictly smaller than one, which by a standard result of matrix analysis implies the spectral radius of $\tilde{A}^{-1}\tilde{B}$ to be less than one (Horn and Johnson (1985)), and the claim follows. Similarly, ii) implies iii) of theorem 1 as the spectral radius of $\partial\tilde{\varphi}(x_1)$ is bounded from above by any matrix norm.

■

If $k = 1$ then it follows from ii) of theorem 1 that there is exactly one symmetric equilibrium if and only if $\tilde{\varphi}(\bar{x})$ crosses the 45°-degree line from above. If theorem 1 cannot be applied, this simple geometric insight provides a constructive way of using the SOFA to establish that only one symmetric equilibrium exists (see sections 4.3 and 4.4 for examples).

3.2 Asymmetric equilibria

If $(x_1, ..., x_N)$ is an asymmetric equilibrium, then a permutation $\left(x_{\sigma(1)}, ..., x_{\sigma(N)}\right)$ gives a further asymmetric equilibrium. The main theorems of this section exploit this symmetry property. We first consider the case of a one-dimensional game. Let $\varphi(x_2; X) \equiv \varphi^1(x_2; X)$ where $X \equiv$

$(x_3, ..., x_N) \in S^{N-2}$, and $\partial\varphi(x_2; X)$ denotes the derivative of $\varphi(\cdot; X)$ with respect to x_2. For given $X \in S^{N-2}$ let

$$T \equiv \{x_2 \in S : \varphi(x_2; X) \in Int(S), \varphi(\cdot; X) \text{ not differentiable in } x_2\}$$

We concentrate on one-dimensional symmetric games satisfying:

$$T = \emptyset \text{ or every } x_2 \in T \text{ is locally isolated} \tag{1}$$

Note that if Π satisfies strong quasiconcavity and differentiability as introduced in section 2, and additionally it is known that $\varphi\left(S^{N-1}\right) \subset Int(S)$, then $T = \emptyset$.

Theorem 2 *Suppose that a one-dimensional symmetric game satisfies* $\varphi(x_{-1}) \in C\left(S^{N-1}, S\right)$ *and* (1) *for any given* $X \in S^{N-2}$. *This game has no asymmetric equilibria if*

$$x_2 \in Int(S)\backslash T, \varphi(x_2; X) \in Int(S) \Rightarrow \partial\varphi(x_2; X) > -1 \tag{2}$$

Proof: While the formal proof can be found in the appendix (7.2), its main idea is illustrated below.

Theorem 2 applies, but is not limited to, games that satisfy the assumptions of section 2. For example, if condition (2) holds for a game with an only piecewise differentiable best-reply function then this game has no asymmetric equilibria. Further, it is noteworthy that while $\varphi(x_{-1}) \in \partial S$ is possible, condition (2) requires to evaluate the slope of φ only at interior points, and if (2) is satisfied, this also rules out asymmetric boundary equilibria. Moreover, the theorem imposes no restrictions on the position or the shape of the best-reply function (up to condition (2)). For example, theorem 2 can be applied to games with non-monotonic behavior (e.g. contests, see section 4.3). Finally, rather than evaluating a $N \times N$-matrix as would be required e.g. by the index theorem, condition (2) needs only information about the behavior of the reply-function in a two-player game (as X can be treated as fixed). Condition (2) is appealing for applied work because, by the IFT, it can be expressed in terms of the second partial derivatives of Π:

Corollary 2 *If in a one-dimensional symmetric game for all* $x_1, x_2 \in Int(S)$ *and any given* $X \in S^{N-2}$ *the condition*

$$\Pi_1(x_1, x_2; X) = 0, x_2 \notin T \quad \Rightarrow \quad \Pi_{11}(x_1, x_2; X) < \Pi_{12}(x_1, x_2; X) \tag{3}$$

is satisfied, then no asymmetric equilibria exist.

It should be mentioned that additional information about $\varphi(x_{-1})$ can further restrict the x_2-range in theorem 2 (or corollary 2). For example, it suffices to verify condition (2) only at points $x_2 \in Int(S) \cap \varphi(S^{N-1})$. I now provide the geometric intuition behind theorem 2 for the simple case where $N = 2$ and $\varphi^{-1}(S) \subset Int(S)$. In essence, it is an application of the Mean Value Theorem, and the idea is illustrated in figure 1. Suppose that the point $A = (x_1^a, x_2^a)$

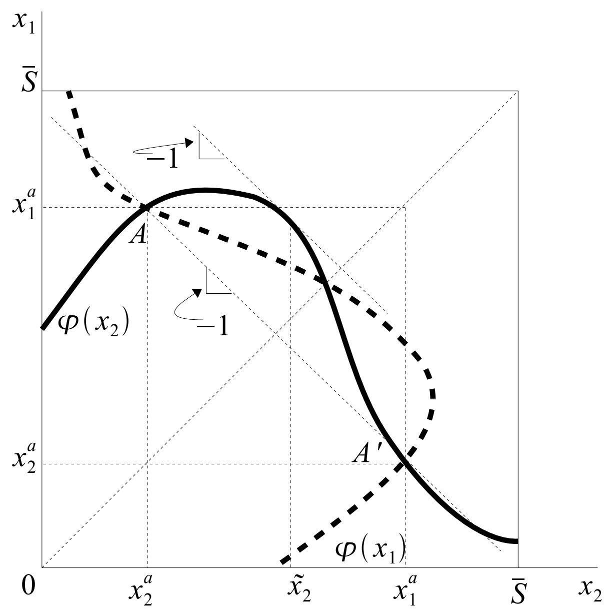


Figure 1: Theorem 2

corresponds to an asymmetric equilibrium. By symmetry its reflection, the point $A' = (x_2^a, x_1^a)$, also is an asymmetric equilibrium. Hence the line that connects A and A' must have a slope of -1. As $\varphi(x_2)$ remains in $Int(S)$, φ is differentiable on $Int(S)$. According to the Mean Value Theorem there is a point $\tilde{x}_2 \in (x_2^a, x_1^a)$ with $\partial\varphi(\tilde{x}_2) = -1$. Hence if in such a game $\partial\varphi(\tilde{x}_2) > -1$ for all $x_2 \in Int(S)$, then there cannot be any asymmetric equilibria.[6]

[6]The general proof (see appendix) is complicated by the fact that $\varphi(x_2)$ is allowed to be on the boundary or not differentiable everywhere, which requires extending the Mean Value Theorem appropriately (see lemmata

I now turn to the higher-dimensional case, and provide an in-depth discussion for $k = 2$. In the appendix (section 7.3) it is shown that the main result (theorem 3) extends beyond $k = 2$. In the two-dimensional case the best-reply of player 1 is a vector-valued function $\varphi(x_2; X) = (\varphi_1(\cdot), \varphi_2(\cdot))$. For notational simplicity I set $(\alpha, \beta, \gamma, \delta) \equiv \left(\frac{\partial \varphi_1}{\partial x_{21}}, \frac{\partial \varphi_1}{\partial x_{22}}, \frac{\partial \varphi_2}{\partial x_{21}}, \frac{\partial \varphi_2}{\partial x_{22}}\right)$, where all partial derivatives are evaluated at $(x_2; X)$.

Theorem 3 below is the two-dimensional extension of theorem 2 for the case where $\varphi(x_2; X) \in Int(S)$ everywhere and, if any, points where $\varphi(x_2; X)$ is not differentiable are locally isolated.

Theorem 3 *Let $k = 2$ and suppose that $\varphi \in C\left(S^{N-1,}S\right)$, $\varphi\left(S^{N-1}\right) \subset Int(S)$, and $\varphi(x_{-1})$ is differentiable except possibly for a set of isolated points. If for all $x_2, x_2' \in S$ and any given $X \in S^{N-2}$ the condition*

$$\alpha(x_2), \delta(x_2') > -1 \qquad (1 + \alpha(x_2))(1 + \delta(x_2')) > \beta(x_2)\gamma(x_2') \tag{4}$$

holds, then no asymmetric equilibria exist.

Proof: Appendix (7.3)

Under the assumptions on Π from section 2, $\varphi(x_2; X) \in Int(S)$ everywhere implies differentiability of $\varphi(x_2; X)$ everywhere[7], which is included in theorem 3 as a special case. Also note that, unlike in the one-dimensional case, $\varphi(x_2; X) \in Int(S)$ everywhere is necessary but not sufficient[8] for the index boundary conditions to be satisfied.

In practice we can use the IFT to express (4) in terms of the second partial derivatives of Π. If $\varphi(x_2; X) \in Int(S)$ then $\partial\varphi(x_2; X) = -H^{-1}B$, where H is the Hessian $\frac{\partial^2 \Pi(x_1, x_2; X)}{\partial x_1 \partial x_1}$ and $B = \frac{\partial^2 \Pi(x_1, x_2; X)}{\partial x_1 \partial x_2}$. It is possible to adapt theorem 3 to the case where $\varphi(x_{-1}) \in \partial S$ may occur. The IFT remains the essential tool to calculate the slopes in applications with boundary solutions, but it must be applied to an extended system of equations. As the central insights

1 and 2, section 7.2).

[7] Follows from strong quasiconcavity and the IFT.

[8] Counterexamples can easily be constructed, see e.g. section 4.4. Hence theorem 3 may be applicable even if the index boundary conditions are violated.

about the existence and properties of asymmetric equilibria remain the same, but the analogue statement to (4) and the proof become messier to write in case of boundary solutions, the result is postponed to the appendix (see remark I in section 7.3).

Theorem 3 sheds light on the nature of asymmetric equilibria in symmetric two-dimensional games. First, as is intuitively clear from $k = 1$, the best-reply function $\varphi_i(x_2; X)$ may not fall to quickly in the i-th component strategy of player two, so suppose that $\alpha, \delta > -1$. Then, interestingly, the cross derivatives β and γ crucially influence whether and what type of asymmetric equilibria may occur in the game. Suppose that $x^a = (x_1^a, x_2^a, ..., x_N^a)$ is an asymmetric equilibrium. I refer to x^a as a strictly ordered equilibrium if $x_g^a > x_h^a$ for any pair of strategies in x_a. I call an equilibrium with $x_{ji}^a > x_{hi}^a$ but $x_{ji'}^a < x_{hi'}^a$ strictly unordered.

Corollary 3 *The following facts are satisfied under the presumptions of theorem 3:*

i) If $\beta(x_2) \geq 0, \alpha(x_2) > -1$ or $\gamma(x_2) \geq 0, \delta(x_2) > -1$, for any $x_2 \in S$ and any given $X \in S^{N-2}$, then there cannot be any strictly ordered equilibria.

ii) If $\beta(x_2) \leq 0, \alpha(x_2) > -1$ or $\gamma(x_2) \leq 0, \delta(x_2) > -1$, for any $x_2 \in S$ and any given $X \in S^{N-2}$, then there cannot be any strictly unordered equilibria.

Proof: Appendix (7.4)

If $\alpha(x_2), \delta(x_2) > -1$ for all $x_2 \in S$ and any given X, the conclusions of corollary 3 extend to weak inequalities. For example, if additionally $\beta \geq 0$, then there cannot be any asymmetric equilibria with $x_g^a \geq x_h^a$. A direct consequence of this is e.g. that games with weakly increasing best-replies can have only (pairwise) strictly unordered asymmetric equilibria, whereas a game with weakly decreasing best-replies (and $\alpha, \delta > -1$) can only have strictly ordered asymmetric equilibria. Finally, provided that $\alpha, \delta > -1$, a game with both partially increasing and decreasing replies (e.g. $\beta \geq 0$ and $\gamma \leq 0$) can *never* have any asymmetric equilibria.

A compact way of expressing (4) is to say that, for any given $X \in S^{N-2}$, the matrix

$$\begin{pmatrix} 1+\alpha(x_2) & \beta(x_2) \\ \gamma(x_2{}') & 1+\delta(x_2{}') \end{pmatrix} = I + \underbrace{\begin{pmatrix} \partial\varphi_1(x_2) \\ \partial\varphi_2(x_2{}') \end{pmatrix}}_{\equiv A(x_2, x_2{}')} \tag{5}$$

has only positive principal minors for $x_2, x'_2 \in S$. Moreover, if φ is everywhere differentiable, (4) can be reduced to the requirement that, for given $X \in S^{N-2}$, every principal minor of (5) is non-zero for any $x_2, x'_2 \in Int(S)$ (see section 7.3). From (5) we can derive further sufficient conditions to rule out asymmetric equilibria. For example, if $\alpha, \delta > -1$ and the spectral radius of the matrix $A(x_2, x'_2)$ is less than one, no asymmetric equilibrium exists. Put differently, if for any (x_2, x'_2) we have $\alpha(x_2), \delta(x'_2) > -1$ and there is a matrix norm $\|\cdot\|$ such that $\|A(x_2, x'_2)\| < 1$, then there cannot be any asymmetric equilibria.

Finally, one can use the IFT to show that if for any $x_1, x_2 \in S$ and any given $X \in S^{N-2}$ the local diagonal dominance condition

$$\Pi_1(x_1, x_2; X) = 0 \text{ or } \Pi_2(x_1, x_2; X) = 0 \Rightarrow |\Pi_{ii}| > \sum_{j \neq i, j \leq 4} |\Pi_{ij}| \qquad i = 1, 2$$

holds, then (4) is satisfied[9], i.e. such a game cannot have any asymmetric equilibria.

3.3 Summary

If the conditions of theorem 1 and of theorem 2 (for $k = 1$) or theorem 3 (for $k = 2$) are satisfied, then the game only has one equilibrium, the interior symmetric equilibrium. Compared to the necessity of evaluating the determinant of a $Nk \times Nk$ matrix as would be required by the univalence or index theorem, the separation approach enables us to reduce the dimensionality of the problem from Nk to k, and allows us to learn more about the nature of equilibria in particular games. This is generally not possible with standard approaches to uniqueness. For example, even if ∇F satisfies the index conditions, and critical symmetric points have an algebraic index sum of $+1$, we may *not* conclude that there are no asymmetric equilibria, as there still could be an even number of asymmetric equilibria. Similarly, an index sum of -1 from critical symmetric points does not necessarily imply the existence of multiple symmetric equilibria (see section 7.1 for more on what possibly could be inferred from the index theorem). Moreover, the index conditions may be violated e.g. because $\varphi \in \partial S$, which is not unusual for many interesting applications (e.g. the contest in section 4.3 or the information-pricing

[9]Using the boundary version of theorem 3 (see section 7.3), it can be shown that this condition also rules out asymmetric equilibria if $\varphi(x_2; X) \in \partial S$ is possible.

game in section 4.4). Theorems 2 and 3 can potentially be applied to non-index games to rule out asymmetric equilibria. Finally, even if $\nabla\tilde{\Pi}$ does not satisfy the index conditions, we may still make use of the SOFA to rule out multiple symmetric equilibria, which is repeatedly demonstrated by several examples in the next section.

4 Applications

First, I show that there is an intimate link between the existence of asymmetric equilibria in a symmetric game and the equilibrium set of asymmetric variations of that game. Second, it is proved that uniqueness in symmetric games regularly extends to uniqueness in almost symmetric games. Finally, the separation approach is applied to several examples.

4.1 Equilibria in asymmetric games

Let $c_g \in \mathcal{P}$ denote player g's parameter vector, where $\mathcal{P} \subset \mathbb{R}^m$ is a compact parameter space. $\Gamma(c) \equiv \left(N, S^N, \{\Pi^g(x, c_g)\}_{g=1}^N\right)$, $c \in \mathcal{P}^N$, is a game[10] with parameters $c_1, ..., c_N$. If $c_1 = c_2 = ... = c_N$ the game is symmetric. For now, we concentrate on one-dimensional games where the heterogeneity of the payoff-functions is restricted to the distribution of one parameter. The following proposition shows that if for a game, where best-replies are increasing in the parameter $c \in [\underline{c}, \bar{c}]$, any underlying two-person symmetric game does not have an asymmetric equilibrium, then the strategies in every equilibrium of the asymmetric game are ordered exactly in the same way as the parameters c_i.

Proposition 3 *Suppose that $\varphi(x_{-1}, c)$ is increasing in c on $[\underline{c}, \bar{c}]$ and $\bar{c} \geq c_1 > c_2, ..., > c_N \geq \underline{c}$. If for any given $X \in S^{N-2}$ and any $c \in [\underline{c}, \bar{c}]$ the symmetric two-player game with payoffs $\Pi^j(x_1, x_2; X, c)$, $j = 1, 2$, has no asymmetric equilibria, then every equilibrium of the asymmetric game $\Gamma(c_1, ..., c_N)$ satisfies $x_1^* \geq x_2^* \geq, ..., \geq x_N^*$. Moreover, $x_1^* > x_2^* >, ..., > x_N^*$ results if $\varphi(x_{-1}, c)$ is strictly increasing in c on $[\underline{c}, \bar{c}]$.*

[10] In this and the next section I assume that $\Pi^j(x, c)$ is twice continuously differentiable in (x, c) and strongly quasiconcave in x_j for any $c \in \mathcal{P}$.

Proof: To prove this proposition we require inter alia a characterization result for asymmetric equilibria (see appendix 7.5). If the game is decreasing in c on $[\underline{c}, \bar{c}]$, the inequalities of the equilibrium strategies are reverted.

Proposition 3 tells us e.g. that asymmetric games never possess symmetric equilibria if φ^j is strictly monotonic in c on $[\underline{c}, \bar{c}]$. Notably, we can use the simple slope condition of theorem 2 to exclude the possibility of equilibria in c-monotonic asymmetric games, which do not reflect the order of the parameters. Proposition 3 extends to the case where c_j is a parameter vector: If $c_1, ..., c_N$ are parameter vectors such that $\varphi(x_{-1}, c_g) \geq \varphi(x_{-1}, c_j)$ and the respective symmetric two-player games have no asymmetric equilibria for any of these parameter vectors, then $x_1 \geq x_2 \geq ... \geq x_N$ holds in any equilibrium of the asymmetric game.

4.2 Uniqueness in almost symmetric games

Proposition 3 shows that we can learn certain properties of the equilibrium set of asymmetric games by studying certain symmetric games. A related question is whether uniqueness in a symmetric game is a property that extends at least to almost symmetric games, i.e. games where the ex-ante asymmetries are small. The answer to this question is definitely yes (for $k \geq 1$), provided that the symmetric equilibrium is regular[11].

Proposition 4 *Suppose that the joint best-reply satisfies $\phi(\cdot, \cdot) \in C\left(S^N \times \mathcal{P}^N, S^N\right)$ and consider a symmetric game $\Gamma(c)$ with a unique, symmetric and regular equilibrium $x^* \in Int(S^N)$. Then $\exists\ \delta > 0$ such that $\Gamma(c')$ has a unique equilibrium for any $c' \in \mathbb{B}(c, \delta)$.*

Proof: Appendix (7.5)

If $k = 1$ and the variation in parameters $c \to c'$ is small and of a c-monotonic nature, propositions 3 and 4 tell us that there is a unique interior equilibrium, and this equilibrium reflects the order of the parameters.

[11] $Det(J(x)) \neq 0$, where $J(x)$ is the Jacobian of $\nabla F(x)$.

4.3 One-dimensional sum-aggregative games

Several interesting games have the property that the strategies enter the payoff functions as a sum. Payoff functions of such sum-aggregative games can be represented[12] as $\Pi(x_g, x_{-g}) = \hat{\Pi}(x_g, Q)$, with $Q = \sum_{j=1}^{N} x_j$.

Proposition 5 *Consider a sum-aggregative symmetric one-dimensional game.*

i) If for $(x_1, Q) \in Int(S) \times (0, \bar{S}N)$ condition

$$\hat{\Pi}_1(x_1, Q) + \hat{\Pi}_2(x_1, Q) = 0 \quad \Rightarrow \quad \hat{\Pi}_{11}(x_1, Q) + \hat{\Pi}_{12}(x_1, Q) < 0 \tag{6}$$

is satisfied, then no asymmetric equilibrium exists.

ii) A sum-aggregative symmetric index game has only one symmetric equilibrium iff the following condition holds on Cr^s:

$$\hat{\Pi}_{11}(x_1, Nx_1) + (N+1)\hat{\Pi}_{12}(x_1, Nx_1) + N\hat{\Pi}_{22}(x_1, Nx_1) < 0 \tag{7}$$

Proof: (i) Use (3) of corollary 2 to obtain (6). (ii) Apply i) of theorem 1 to obtain (7).

■

Example 1*: Cournot*

The symmetric Cournot model has $\hat{\Pi}(x_1, Q) = P(Q)x_1 - c(x_1)$, where Q is the aggregate quantity supplied, P is inverse market demand and $c(x_1)$ are quantity costs. Presuming that the symmetric index conditions are satisfied, there is exactly one symmetric Cournot equilibrium iff

$$P(Nx_1) + P'(Nx_1)x_1 - c'(x_1) = 0 \quad \Rightarrow \quad N\left(P'(Nx_1) + P''(Nx_1)x_1\right) < c''(x_1) - P'(Nx_1) \tag{8}$$

Moreover, from (6) we deduce that if $P' < c''$ is satisfied (whenever $P(Q) - c'(x_1) + P'(Q)x_1 = 0$), then the Cournot game has no asymmetric equilibria. Kolstad and Mathiesen (1987) derive

[12] Note that e.g. a game with payoff $\Pi(x_g, \sum f(x_j))$, where $f \in C^2(S, \mathbb{R})$ is strictly increasing, can be equivalently represented as a sum-aggregative game using the change of variable $e_j = f(x_j)$.

general conditions of uniqueness for the (non-symmetric) Cournot game, imposing $P' < c''$ as an exogenous assumption. While this is reasonable on intuitive grounds, we learn from the separation approach that exactly this assumption rules out the possibility of asymmetric equilibria - and *therefore* is a natural precondition for uniqueness. As a consequence we see that non-uniqueness of equilibria in the symmetric (or almost symmetric) Cournot model arises mainly from the possibility of multiple symmetric equilibria and not from asymmetric equilibria. Moreover, if $P' < c''$ and there is a unique equilibrium in a symmetric Cournot index game, the equilibrium is stable (see section 5). It should be mentioned that $P' < c''$ also rules out the possibility of asymmetric equilibria even if $\varphi(x_{-1}) \in \partial S$ or $\varphi(x_{-1})$ has isolated kinks[13], which is not unrealistic for the Cournot model as market demand may have kinks e.g. because of heterogeneous consumers. Note that the symmetric index theorem can be applied to rule out multiple symmetric equilibria even if $\tilde{\varphi}$ has kinks, provided that the index conditions are satisfied (i.e. kinks are not symmetric equilibria). If the uniqueness-condition[14] of Kolstad and Mathiesen (1987) is evaluated for the symmetric case under the assumption that $P' < c''$ we obtain *exactly* condition (8) ruling out multiple symmetric equilibria - which besides the simplicity of obtaining the result also illustrates the generality of the separation approach.

Example 2: Contests

Consider a general sum-aggregative contest $\Pi = p\left(g(y_1), \sum_{j=1}^{N} g(y_j)\right) V - h(y_1)$, where $V, g' > 0$ and $p \in [0,1]$ is a contest success function (see Konrad (2009)). Note that, by a change of variables, such a contest may be represented as $\hat{\Pi} = p(x_1, Q)V - c(x_1)$, where $c(x_1) = h\left(g^{-1}(x_1)\right) \in C^2$. Using (6), (7) and assuming that the symmetric index conditions are satisfied, we may conclude that such a contest has a unique symmetric equilibrium if at corresponding critical points:

$$(p_{11}(x_1, Q) + p_{12}(x_1, Q))V - c''(x_1) < 0 \tag{6'}$$

$$(p_{11}(x_1, Nx_1) + (N+1)p_{12}(x_1, Nx_1) + Np_{22}(x_1, Nx_1))\, V - c''(x_1) < 0 \tag{7'}$$

[13]In such cases the index theorem obviously is not applicable.

[14]Corollary 3.1, p. 687

Suppose that $c(0) = c'(0) = 0$ and

$$p(x_1, Q) = \begin{cases} \frac{1}{N+r} & x_1 = ... = x_N = 0 \\ \frac{1}{1+r} & x_1 > 0, x_2 = ... = x_N = 0 \\ f\left(\frac{x_1}{Q+r}\right) & \text{else} \end{cases}$$

where $r \geq 0$, $f \in C^2$ is strictly increasing, concave and $f'(0) > 0$. The best-reply $\varphi(x_{-1}) \in (0, \bar{S}]$ is continuous, and differentiable if $\varphi(x_{-1}) \in Int(S)$ whenever $x_2 > 0$. It can be verified that (6') is satisfied, meaning that there cannot be any asymmetric contest equilibria. Turning to symmetric equilibria we note that $x_1 = 0$ can never be a best-reply to any $x_{-1} \in S^{N-1}$. While we cannot use the (symmetric) index theorem because Π is not differentiable at the origin, it is straightforward to verify that this example satisfies (7') for respective interior points. As (7') implies that $\tilde{\varphi}(\bar{x})$ can cross the 45°-line at most once on $(0, \bar{S}]$, we conclude that there is a unique symmetric equilibrium $x_1^* \in (0, \bar{S}]$. If $f(z) = z$ then the previous example collapses to the often invoked Tullock success function. Hence the separation approach also provides us with a simple proof of uniqueness for the Tullock contest.

Example 3: Homogeneous revenue

The Tullock contest with $r = 0$ is an important example, where revenues are homogeneous functions. Applying the separation approach to sum-aggregative games with homogeneous revenues and strictly convex costs reveals that such games *naturally* have only one symmetric equilibrium, which also very likely is the unique equilibrium of the game. To see this consider $\Pi(x) = \pi(x_1, \sum x_j) - c(x_1)$, where $\pi(x_1, \sum x_j)$ is homogeneous of degree $z < 1$ in $(x_1, ..., x_N)$, or equivalently $\pi(x_1, Q)$ is z-homogeneous in (x_1, Q).

Proposition 6 *Suppose that $\pi(x_1, Q)$ is homogeneous of degree $z < 1$ in (x_1, Q) and $c', c'' > 0$. Then there is only one symmetric equilibrium. If additionally, $\pi_1 \geq 0$ and $\pi_{11} \leq 0$ for $x_1 > 0$ the symmetric equilibrium is unique.*

Proof: We start with the second claim. As $\pi_1(x_1, Q) \geq 0$ is $z-1$-homogeneous the Euler-theorem and the sum-aggregative structure imply that $\pi_{11} + \frac{Q}{x_1}\pi_{12} \leq 0$ for $x_1 > 0$, which if $\pi_{11} \leq 0$ necessarily implies that $\pi_{11} + \pi_{12} \leq 0$. As $c''(x_1) > 0$ for $x_1 > 0$ there cannot be any asymmetric equilibria by (6). Turning to symmetric equilibria, as $\frac{\partial \Pi(x_1, \sum x_j)}{\partial x_1}\Big|_{x_j=x_1}$ is $(z-1)$-homogeneous in x_1 we must have that $\nabla\tilde{\Pi}(x_1) = \omega x_1^{z-1} - c'(x_1)$, where $\omega > 0$ is a constant. Hence $\tilde{J}(x_1) = \omega(z-1)x_1^{z-2} - c''(x_1) < 0$ whenever $x_1 > 0$, which implies that $\tilde{\varphi}(\bar{x})$ can intersect the 45°-line at most once. Thus there cannot be multiple symmetric equilibria.

■

Proposition 6 is another way of proving uniqueness in the symmetric Tullock contest (for $r = 0$) or the Cournot model with inverse demand $P(Q) = Q^{-\alpha}$, $\alpha \in (0, 1]$.

4.4 Information-pricing game

In this section I apply the separation approach to a two-dimensional information-pricing game as introduced by Grossman and Shapiro (1984). Each of two firm chooses its price p and the market fraction a of consumers to be made aware of its product, taking $(\bar{p}, \bar{a})$ of its opponent as given. There is a measure of δ consumers, ex-ante unaware of both firms. Ex-post a consumer might be aware of none, of both or of just one firm. Information (advertisement) is distributed randomly and independently over the population, firms cannot discriminate between consumers, and products are imperfect substitutes. A firm's market demand from consumers not aware of the other firm is $x(p)$, and $x(p, \bar{p})$ for consumers that receive ads from both firms. Assuming constant unit costs of production the firm's expected profit is

$$\Pi(p, a) = a\left[(1-\bar{a})\underbrace{(p-c)x(p)}_{\equiv V(p)} + \bar{a}\underbrace{(p-c)x(p,\bar{p})}_{\equiv V(p,\bar{p})}\right]\delta - C(a) \equiv aV(p, \bar{p}, \bar{a})\,\delta - C(a) \qquad (9)$$

where $C(a)$ are information costs. It is reasonable to assume that a firm's demand for given prices $p, \bar{p}$ is never lower if a consumer is not aware of its competitor ($x(p) \geq x(p, \bar{p})$). Similarly, demand very likely reacts more sensitively towards an unilateral price change in case of perfectly informed consumers ($x_p(p, \bar{p}) \leq x'(p)$), and (marginal) demand never decreases in the opponents price ($x_{\bar{p}}, x_{p,\bar{p}} \geq 0$). Intuitively, most of these facts can be justified under free-trade, as perfectly informed consumers have two outside options (not consume or consume at the competitor's location) whereas unilaterally informed consumers just have one (not consume). To be precise, the following formal assumptions are imposed:

The function $V(p,\bar{p},\bar{a}) \in C^2(S^2,\mathbb{R})$, where $S=[c,\hat{p}]\times[0,1]$, is strongly quasiconcave in p, $V_{\bar{a}}, V_{p\bar{a}} \leq 0$, $V_{\bar{p}}, V_{p\bar{p}} \geq 0$ and $\hat{p} > c$ is the monopoly price. The cost function satisfies $C(0) = C'(0) = 0$ and $C'(a), C''(a) > 0$ for $a > 0$. Moreover, it is assumed that $\varphi = (p,a)(\bar{p},\bar{a}) \in Int(S)$ for any $(\bar{p},\bar{a}) \in [c,\hat{p}]\times(0,1]$ (interior best-replies), and that there exists $p \in [c,\hat{p}]$: $V(p,c,1) > 0$. The last assumption means that even under perfect information and marginal-cost pricing of the opponent the firm can retain a strictly positive market demand for a price slightly above marginal costs, which is a typical feature competition with imperfect substitutes. A simple example for V is demand derived from quadratic utility (LaFrance (1985)), i.e. $x(p) = 1-p$ and $x(p,\bar{p}) = \frac{1-p+\gamma(\bar{p}-1)}{1-\gamma^2}$, where the parameter $\gamma \in (0,1)$ controls the degree of substitutability. It is easy to see that e.g. for $c = 0$, $S_p = [0,1/2]$ and $\gamma \in [0,1/2]$ this example satisfies the above assumptions. Despite that (9) looks innocent, investigating the set of equilibria is not trivial. For example, the index theorem cannot be used as the boundary conditions are naturally violated in this model[15]. Moreover, even if it were applicable, we would have to calculate the determinant of a largely abstract 4×4-matrix.

I now use the separation approach to investigate the equilibrium set of this two-dimensional game. We note that because $V(p,\bar{p},\bar{a}) > 0$ is always feasible, $a = 0$ can never be a part of a firm's best reply. Because Π is continuous, V is strongly quasiconcave in p and $C'' > 0$ the best-reply $\varphi = (p,a)$ is a continuous function of $(\bar{p},\bar{a})$. Consequently, at least one symmetric equilibrium exists in this game. Further, the above assumptions imply[16] that $p'(\bar{p}) \geq 0$ and $p'(\bar{a}) \leq 0$. Hence, by corollary 3, we may already conclude that if asymmetric equilibria exist, these equilibria must be (weakly) ordered. As also $a'(\bar{p}) \geq 0$ there cannot be any asymmetric equilibria if $a'(\bar{a}) > -1$ according to corollary 3. This condition is satisfied if for $a, \bar{a} \in (0,1)$ we have that (use the FOC and the IFT):

$$\frac{V(p)-V(p,\bar{p})}{(1-\bar{a})V(p)+\bar{a}V(p,\bar{p})} < \frac{C''(a)}{C'(a)} \tag{10}$$

The LHS of (10) is maximal (for a given p) if $(\bar{p},\bar{a}) = (c,1)$. Hence if e.g. $\frac{V(p)}{V(p,c)} < \frac{C''(a)}{C'(a)} + 1$,

[15]For example, $\Pi_a(c,0) = V(c,\bar{p},\bar{a})\delta - C'(0) = 0$, i.e. ∇F does not point inwards at $(c,0,\bar{p},\bar{a}) \in \partial S^2$.

[16]Follows from applying the IFT to FOC's.

then (10) is satisfied. However, we may exploit the FOC to obtain a better estimate (see below). From the analysis so far we learn two things about the scope of asymmetric equilibria in the information-pricing game. First, the fact that only ordered asymmetric equilibria may exist (if any at all) is independent of scale effects. This can be seen as none of the results obtained so far depends on the market size parameter δ, nor on unit production costs c nor on multiplicative information cost parameters (if $C(a) = \theta c(a)$ then $\theta > 0$ plays no role). Second, according to (10) only if either marginal costs react sufficiently inelastically to a change in advertising, or monopoly rents exceed the duopoly rents by a relatively large amount (e.g. because products are strong substitutes) there could be asymmetric specialization equilibria. Intuitively, in such an equilibrium one firm can be thought of specializing on advertising (high (p, a)) earning quasi-monopoly rents from unilaterally informed consumers but incurring high advertising costs, whereas the other firm specializes in competition (low (p, a)) and thereby wins the good informed consumers, but faces only little demand because of a small advertising campaign. What is the scope for such asymmetric equilibria in our parametric example? Exploiting the linearity of the problem it can be shown that $p(\bar{p}, \bar{a}) = \frac{1-\gamma^2-\gamma(1-\bar{p}-\gamma)\bar{a}}{2-2\gamma^2(1-\bar{a})} \in (\frac{1-\gamma}{2}, \frac{1}{2})$ for $\gamma, \bar{p} \in [0, 1/2]$ and $\bar{a} \in (0, 1]$. Using $\bar{p} = \frac{1-\gamma}{2}$ and $\bar{a} = 1$ in the LHS of (10) reveals that the left-hand side of (10) is smaller than $\frac{\gamma}{1-\gamma-\gamma^2} \leq 2$. Hence, if $\frac{C''(a)}{C'(a)} \geq 2$ we can be sure that no asymmetric equilibrium exists. More concretely, if $C(a) = \theta a^\eta$, $\eta \geq 2$, then no asymmetric equilibrium exists if $\eta \geq 3$ or competition is not too intense (if $\gamma \leq \sqrt{2} - 1$). A similar conclusion holds for the CRIR advertising technology introduced by Grossman and Shapiro (1984): $C(a) = \frac{Ln(1-a)}{Ln(1-r)}$, $r \in (0, 1)$, implies that $\frac{C''(a)}{C'(a)} = \frac{1}{1-a} \geq 1$. Hence if $\gamma \leq \sqrt{2} - 1$ and advertising technology follows the CRIR technology, there cannot be any asymmetric equilibria. All in all we conclude that while the scope for asymmetric equilibria in this game is small.

Turning to the issue of multiple symmetric equilibria, we note that

$$\nabla\tilde{\Pi}(p, a) = \begin{pmatrix} aV_1(p, p, a)\delta \\ V(p, p, a)\delta - C'(a) \end{pmatrix} \tag{11}$$

(11) shows that the index theorem is not applicable even if we restrict attention to symmetric equilibria as $\nabla\tilde{\Pi}$ vanishes e.g. at the corner point $(p, a) = (c, 0)$. Whereas $(c, 0)$ is a zero of (11),

i.e. an equilibrium candidate, it obviously cannot constitute a symmetric equilibrium. While we cannot rely on the index theorem to discuss the scope of multiple symmetric equilibria, (11) provides us with a useful guideline to prove uniqueness in a more constructive way. Assuming that $\nabla\tilde{\Pi}(p,a) = 0$ for some interior point (p,a) we obtain $Det(\tilde{J}(p,a)) > 0$ iff $V_{1p}(V_a\delta - C'') - V_{1a}V_p\delta > 0$. If the index theorem were applicable, we could conclude that if i) $V_1(p,p,a) = 0 \Rightarrow V_{1p}(p,p,a) < 0$ and ii) $V(p,p,a)\delta - C'(a) = 0 \Rightarrow V_p(p,p,a) \geq 0$ then there is exactly one symmetric equilibrium (p,a). The claim now is that these conditions in fact imply this result without invoking the index theorem. To see this, consider the pure symmetric pricing game, where each firm solves $\max_{p_i \in [c,\hat{p}]} V(p_i,p_j,a)\delta$ for given $a > 0$. Then i) assures the existence of a single symmetric equilibrium $p = p(a) \in (c,\hat{p}]$, because $\tilde{p}(\bar{p};a)$ can reach the 45°-line just once. Moreover, $p(0) = \hat{p}$, p is continuous in a and if $p(a) \in (c,\hat{p})$, then $p'(a) = \frac{-V_{13}}{V_{1p}} \leq 0$. Next, consider the pure symmetric information game, where each firm solves $\max_{a_i \in [0,1]} a_iV(p,p,a_j)\delta - C(a_i)$. Then ii) implies the existence of a single symmetric equilibrium $a = a(p) \in [0,1]$, where $a(c) = 0$ and a is continuous in p. If $a(p) \in (0,1)$, then $a'(p) = \frac{V_p\delta}{V_3\delta - C''} \geq 0$. A symmetric equilibrium of the original information-pricing game is a FP of the mapping $(p(a), a(p))$, and the above analysis shows that there is exactly one such FP (see figure 2). Note that the important

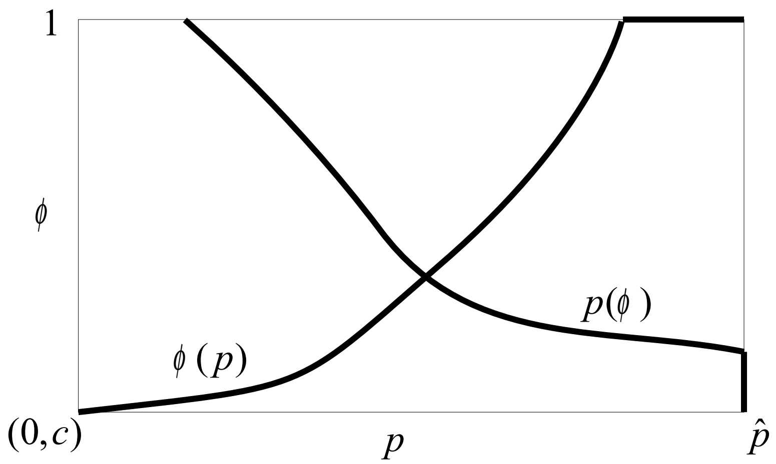


Figure 2: Inexistence of multiple symmetric equilibria

part of the above derivation is that $p'(a) \leq 0$ but $a'(p) \geq 0$ hold at interior points, which essentially reflects the nature of the strategies in this game. Therefore, we conclude that the scope for multiple symmetric equilibria is small and, overall, uniqueness of equilibria is a rather likely outcome in the information-pricing game. In particular, our parametric example satisfies all above conditions and therefore has a only one symmetric equilibrium. To summarize:

Proposition 7 *In the information-pricing game with linear demand there is a single symmetric equilibrium. For $C(a) = \theta a^\eta$, $\eta \geq 2$, the symmetric equilibrium is unique if information costs are sufficiently elastic ($\eta \geq 3$) or products are not too strong substitutes ($\gamma \leq \sqrt{2} - 1$).*

5 Stability of symmetric equilibria

In this section I investigate the connection between symmetric stability and symmetric equilibria. I show that stability under symmetric adjustments implies the existence of only one symmetric equilibria for $k \geq 1$, and is equivalent to the existence of only one symmetric equilibrium for one-dimensional games. Besides the self-interest of such a formal equivalence, there is a strong link between stability and the comparative statics of a symmetric equilibrium. Consider the system of gradient dynamics

$$\dot{x}_{ji} = s_i \frac{\partial \Pi^j(x)}{\partial x_{ji}} \qquad 1 \leq j \leq N, \quad 1 \leq i \leq k \tag{12}$$

where $s_i > 0$ is an arbitrary rate of adjustment (Dixit (1986)). A solution to (12) has the form $x(t) = (x_j(t))_{1 \leq j \leq N}$, where $x_j(t) = (x_{j1}(t), ..., x_{jk}(t))$ is the vector trajectory of player j. An equilibrium x^* is (asymptotically) stable if all eigenvalues of $\breve{J}(x^*)$, the Jacobian corresponding to system (12), have negative real parts. When considering the comparative statics of symmetric equilibria it makes sense to consider a restricted version of that trajectory map, where the initial values $x(0)$ are required to be symmetric, i.e. $x_1(0) = ... = x_N(0)$. Then, by symmetry, the time path $x_j(t)$ must be the same for all players and solves

$$\dot{x}_{1i} = s_i \tilde{\Pi}_i(x_1) \qquad 1 \leq i \leq k \tag{13}$$

where $\tilde{\Pi}_i(x_1)$ is the i-th projection of $\nabla \tilde{\Pi}(x_1)$. Let $\hat{J}(x_1)$ denote the Jacobian corresponding to (13) and suppose that x^* is an interior symmetric equilibrium. I call x^* symmetrically stable if all eigenvalues of $\hat{J}(x_1^*)$ have negative real parts. If x^* is symmetrically stable then $\lim_{t \to \infty} x(t) = x^*$ if $x(0)$ is symmetric and close to x^*. As figure 3 illustrates, stability of (12) always implies symmetric stability, but not vice-versa. The reason for this essentially is that

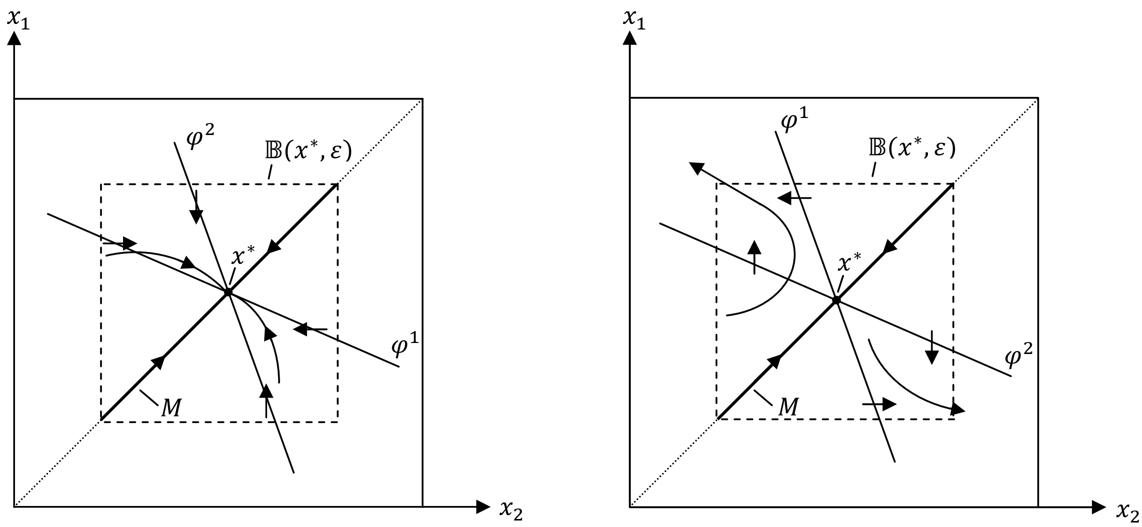


Figure 3: Stable eq. (left), only symmetrically stable eq. (right)

symmetric stability is a special case of the saddle-path theorem.[17] The following proposition reveals the relationship between symmetric stability and (non)-multiplicity of symmetric equilibria.

Proposition 8 *Consider a symmetric game.*

i) If $k = 1$ then there are symmetrically unstable equilibria iff there are multiple symmetric equilibria in a symmetric index game. Provided that x is the only symmetric equilibrium, x is stable under (12) *iff* $\Pi_{11}(x) < \Pi_{12}(x)$.

ii) For $k = 2$ a symmetric equilibrium x is symmetrically stable, if $-\tilde{J}(x_1)$ has only positive principal minors.

iii) If $k \geq 1$ and multiple symmetric equilibria exist, then there are symmetrically unstable equilibria in a symmetric index game.

<u>Proof</u>: Appendix (7.7)

If $k = 1$ and a symmetric index game has only one symmetric equilibrium, this equilibrium is even globally stable for symmetric initial conditions. Several sufficient conditions for symmetric

[17]The spectrum of $\hat{J}(x^*)$ belongs to the spectrum of $\breve{J}(x^*)$. But if the spectrum of $\hat{J}(x^*)$ consists only of eigenvalues with negative real part, then $\breve{J}(x^*)$ must have (at least) k eigenvalues with negative real part. Therefore, by the saddle-path theorem, there exists a k-dimensional manifold M about x^* on which $x(t)$ converges to x^*.

stability can be derived for $k \geq 1$, see section 7.8 of the appendix. Note that if in a symmetric index game any of these conditions are satisfied on Cr^s then, by iii), there is a single symmetric equilibrium, and it is symmetrically stable. Using ii) of proposition 8 it can be verified that our assumptions asserting the existence of only one symmetric equilibrium (p, a) in the information-pricing game (section 4.4) also imply (p, a) to be symmetrically stable (provided it is interior). From i) of proposition 8 it is only a small step to recognize that:

Corollary 4 *If a one-dimensional sum-aggregative index game satisfies conditions* (6) *and* (7)*, then the unique symmetric equilibrium is stable under* (12)*.*

Thus if the Cournot model satisfies $P' - c'' < 0$ as well as the symmetric index conditions and has a unique equilibrium, this equilibrium naturally is stable under (12), which replicates a result of Dastidar (2000) as a special case of corollary 4. If a game has symmetrically unstable equilibria (e.g. because of multiple symmetric equilibria) the comparative statics become problematic. To see why suppose e.g. that a one-dimensional symmetric index game has three asymmetric equilibria $x^A(c), x^B(c), x^C(c)$, where c is an exogenous parameter vector. Such a situation is depicted in figure 4, where A and B are symmetrically stable equilibria (index +1), but C is symmetrically unstable (index −1). Consider a symmetric parameter shift $c \to c'$ and assume that $\tilde{\varphi}(\bar{x}, c') > \tilde{\varphi}(\bar{x}, c)$. The arrows in figure 4 correspond to the dynamics under parameter constellation c'. As is suggested by the figure (formally we would apply the IFT) we see that

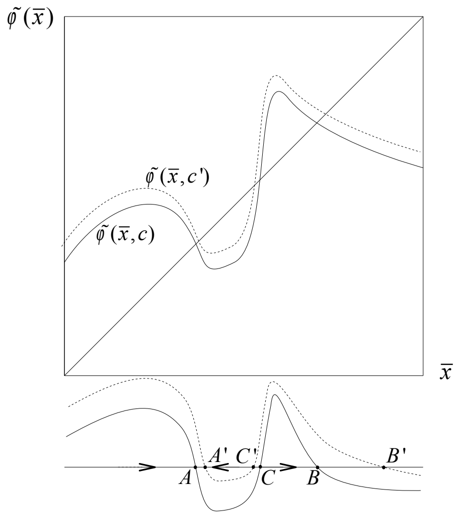


Figure 4: Symmetric stability and comparative statics

the points A and B both increase to A' and B'. As both A' and B' are symmetrically stable, the symmetric dynamics (13) converge from A to A' or from B to B'. For symmetrically unstable points things are different. First, we see that $C' < C$ (a consequence of the negative index at C), contradicting the direction suggested by the monotonicity of the exogenous change. Second, C lies in the basin of attraction of B', which means that the dynamics do not move down to C' but monotonically up to B' (which is not a "small" distance). Hence the gradient dynamics and the comparative-static shift of the equilibrium disagree at C, which helps to explain why one could regard the prediction $C \to C'$ as counterintuitive and not plausible. Notably, such an outcome could never be supported as a stable equilibrium and necessarily requires strong local strategic complements ($\Pi_{12}(x)(N-1) > -\Pi_{11}(x)$ or equivalently $\tilde{\varphi}'(x_1) > 1$).
Summarizing, we see that there is a strong link between symmetric stability and the multiplicity of symmetric equilibria. Further, while multiple symmetric equilibria might impose a problem for comparative static analysis, the existence of asymmetric equilibria is less problematic as long as symmetric shocks are considered.

6 Conclusion

This article exploits the symmetric structure of symmetric games to derive comparably simple tools to investigate the equilibrium set of such games by separating between the possibility of multiple symmetric equilibria and asymmetric equilibria. The practical and theoretical usefulness of this approach was demonstrated with several examples. Compared to other standard methods, in particular the index theorem, the strength of the separation approach are i) its relative simplicity, as the complexity of the fixpoint problem is essentially reduced to a two-player game, ii) the fact that boundary conditions are far less troublesome and iii) best-replies may have kinks. As the application to the two-dimensional price-information game illustrated, the separation method allows to systematically investigate the equilibrium set of higher-dimensional games, which is important for understanding many real-world problems that have eluded a formal assessment so far. For example, one can examine how the nature of strategies in a symmetric game or certain parameter constellations influence the possibility of multiple symmetric or asymmetric equilibria. Finally, analyzing the equilibrium set of a symmetric game may not only be of self-interest, but also sheds light on the equilibria of asymmetric variations of the game, which was illustrated by means of monotonic parameter shifts in one-dimensional games. All in all we believe that our results will provide valuable guidelines for a thorough equilibrium analysis of complex symmetric games in future applied research in game theory, industrial economics and other related fields.

7 Appendix

7.1 A counting rule

If asymmetric equilibria exist, the set of asymmetric equilibria which are permutations of each other with respect to $\{1, ..., N\}$ form an equivalence class within the set of all asymmetric equilibria. I refer to a class of equivalent asymmetric equilibria simply as an equivalent asymmetric equilibrium. Consider a symmetric game, where the C^1-vector field ∇F satisfies i) ∇F has only regular zeroes and ii) ∇F points inwards on the boundary of S^N. Let $\mathcal{I}^s$ denote the sum of the indices of all symmetric equilibria with respect to ∇F.

Proposition 9 *Consider a symmetric game, where ∇F satisfies the above index conditions.*

(a) If $\mathcal{I}^s = 1$ and there are asymmetric equilibria, then there is more than one equivalent asymmetric equilibrium. If especially $N = 2$ then there is an even number of equivalent asymmetric equilibria.

(b) If $\mathcal{I}^s \neq 1$ then asymmetric equilibria exist. For $N = 2$:

(i) if $\mathcal{I}^s = 3 + 4z$ for $z \in \mathbb{Z}$ then there is an odd number of equivalent asymmetric equilibria

(ii) if $\mathcal{I}^s = 5+4z$ for $z \in \mathbb{Z} \setminus \{-1\}$ then there is an even number of equivalent asymmetric equilibria

Proof: Let $\omega \geq 1$ be the (necessarily odd) number of symmetric equilibria. Hence $\mathcal{I}^s$ must be a number from $\{\pm 1, \pm 3, \pm 5, ..., \pm \omega\}$. Further, if $\mathcal{I}^a$ denotes the index sum of all asymmetric equilibria, we must have $\mathcal{I}^s + \mathcal{I}^a = 1$. Note that all asymmetric equilibria in a given equivalence class have the same index. If $\mathcal{I}^s = 1$ but there are asymmetric equilibria, then $\mathcal{I}^a = 0$, which requires the existence of at least two equivalent asymmetric equilibria. If $\mathcal{I}^s \neq 1$ we must have $\mathcal{I}^a \neq 0$, which implies the existence of asymmetric equilibria. To see the rest set $N = 2$ and note that, if asymmetric equilibria exist, there are exactly two asymmetric equilibria within an equivalence class. Let n_- denote the number of equivalence classes with index -1, and n_+ those with index $+1$. Then $n_+ - n_- = \frac{1-\mathcal{I}^s}{2}$. If $\mathcal{I}^s$ is a number $3 + 4z$, the RHS of this equation is an odd number. Hence either n_- or n_+ must be odd and the other number must be even or zero. Consequently, $n_- + n_+$ must be odd. For $\mathcal{I}^s = 5 + 4z$ with $z \in \mathbb{Z} \backslash \{-1\}$ the RHS must be even and hence $n_1 + n_2$ must also be even. Finally, if $\mathcal{I}^s = 1$ then $n_- = n_+ = n$. For $n > 0$ this implies $n_- + n_+ = 2n$, which is even.

■

7.2 Proof of theorem 2

The proofs of theorems 2 and 3 build on the following lemmata.

Lemma 1 *Let $\psi \in C([t_0, t_1], [a, b])$ with $\psi(t_0) \neq \psi(t_1)$. Suppose that the points in (t_0, t_1) where $\psi(t)$ is not differentiable are locally isolated. Then*

$$\begin{array}{ll} i) & \text{if } \psi(t_0) > \psi(t_1)\ \exists t' \in (t_0, t_1) \text{ such that } \psi'(t') \leq \frac{\psi(t_1)-\psi(t_0)}{t_1 - t_0} \\ ii) & \text{if } \psi(t_0) < \psi(t_1)\ \exists t'' \in (t_0, t_1) \text{ such that } \psi'(t'') \geq \frac{\psi(t_1)-\psi(t_0)}{t_1 - t_0} \end{array} \tag{14}$$

<u>Proof</u>: $\tilde{A} \subset (t_0, t_1)$ is the set of non-differentiable points of ψ and $A = \tilde{A} \cup \{t_0, t_1\}$. Let $\psi(t_0) > \psi(t_1)$, define $g(t) \equiv \frac{\psi(t_0)-\psi(t_1)}{t_0 - t_1}(t - t_0) + \psi(t_0)$ and $k(t) \equiv \psi(t) - g(t)$ for $t \in [t_0, t_1]$. Hence $k(t_0) = k(t_1) = 0$, k is continuous on $[t_0, t_1]$ and differentiable at t if $t \notin A$. The proof is by contradiction. Suppose that $\psi'(t) > \frac{\psi(t_1)-\psi(t_0)}{t_1 - t_0}$ holds, whenever $\psi(t)$ is differentiable. If $\tilde{A} = \emptyset$ then k is strictly increasing on $[t_0, t_1]$ by the Mean Value Theorem (MVT), which contradicts $k(t_0) = k(t_1)$. Hence suppose that $\tilde{A} \neq \emptyset$. Then, by local isolation, $\forall t \in \tilde{A}$ there is an interval $I_t = (t - \varepsilon_1, t + \varepsilon_2)$ such that k is differentiable on $I_t \backslash \{t\}$. In fact we can choose $\varepsilon_2 > 0$ such that $t + \varepsilon_2 \in A$. Then the MVT and continuity of k at t imply k to be strictly increasing

over I_t. As for any $t \in \tilde{A}$ $\exists$ $q(t) \in \mathbb{Q} \cap I_t$, the mapping $q : \tilde{A} \to \mathbb{Q}$ is well-defined and injective, which shows that $\tilde{A}$ is countable. Hence we can find a sequence (q_n) with $q_n \in \mathbb{Q} \cap (t_0, t_1)$ such that $q_n \to t_1$ and $k(q_{n+1}) > k(q_n)$, which implies that $k(t_1) > k(q_0)$ by the continuity of k. As $k(t_1) = 0$ we conclude that $k(q_0) < 0$. For exactly the same reason we can also find a strictly decreasing sequence $\tilde{q}_n$, where $\tilde{q}_0 = q_0$, $\tilde{q}_n \to t_0$ and $k(\tilde{q}_{n+1}) < k(\tilde{q}_n)$. Then continuity and the fact that $k(\tilde{q}_0) < 0$ imply $k(t_0) < 0$, a contradiction. This proves i), and ii) follows from i) by setting $\rho(t) \equiv \psi(t_0 + t_1 - t)$.

■

Lemma 2 *Let* $\psi \in C([t_0, t_1], [a, b])$ *with* $\psi(t_0) \neq \psi(t_1)$ *and* ψ *differentiable on* $\psi^{-1}((a, b))$ *except possibly at a set of isolated points. Then* (14) *is satisfied.*

Proof: By the proof of lemma 1 it suffices to consider the case $\psi(t_0) > \psi(t_1)$. Hence $\psi(t_0) > a$ and $\psi(t_1) < b$. Let $T \equiv \psi^{-1}(\{a, b\}) \subset [t_0, t_1]$. If $T = \emptyset$ then the claim follows from lemma 1, so suppose that $T \neq \emptyset$. Note that T is a compact subset of $\mathbb{R}$, and let the min and max of T be denoted by $\underline{t}, \bar{t}$. The proof now is case-by-case. (I) $\psi(\underline{t}) = a$. Then ψ is continuous on $[t_0, \underline{t}]$ and differentiable on $(t_0, \underline{t})$ except possibly for a set of isolated points. Then because of lemma 1 $\exists t \in (t_0, \underline{t})$ such that $\psi'(t) \leq \frac{a-\psi(t_0)}{\underline{t}-t_0} \leq \frac{\psi(t_1)-\psi(t_0)}{t_1-t_0}$. (II) $\psi(\bar{t}) = b$. Then ψ is continuous on $[\bar{t}, t_1]$ and differentiable on $(\bar{t}, t_1)$ except possibly for a set of isolated points. Thus, by lemma 1, $\exists t \in (\bar{t}, t_1)$ such that $\psi'(t) \leq \frac{\psi(t_1)-b}{t_1-\bar{t}} \leq \frac{\psi(t_1)-\psi(t_0)}{t_1-t_0}$. (III) $\psi(\underline{t}) = b$ and $\psi(\bar{t}) = a$. Define $A \equiv \psi^{-1}(\{b\})$, which is a non-empty and compact set. Hence $\hat{t} = max\, A$ exists. Similarly, $B \equiv [\hat{t}, t_1] \cap \psi^{-1}(\{a\})$ also is non-empty and compact. Let $\check{t} = min\, B$. Hence ψ is continuous on $[\hat{t}, \check{t}]$ and differentiable on $(\hat{t}, \check{t})$ except possibly for a set of isolated points. Thus, by lemma 1, $\exists t \in (\hat{t}, \check{t})$ such that $\psi'(t) \leq \frac{a-b}{\hat{t}-\check{t}} \leq \frac{\psi(t_1)-\psi(t_0)}{t_1-t_0}$.

■

Proof of theorem 2

Step 1: $N = 2$. Suppose that (x_1^a, x_2^a) is an asymmetric equilibrium. Then (x_2^a, x_1^a) is a different asymmetric equilibrium and $\varphi(x_2^a) = x_1^a$ and $\varphi(x_1^a) = x_2^a$. Let $\psi(t) \equiv \varphi(x_1^a + t(x_2^a - x_1^a))$ for $t \in [0, 1]$. Then $\psi(0) = x_2^a$ and $\psi(1) = x_1^a$. Hence $\psi \in C([0, 1], S)$, $\psi(0) \neq \psi(1)$ and $\psi(t)$ is differentiable whenever $\psi(t) \in Int(S)$ except possibly for a set of isolated points. If $\psi(0) > \psi(1)$, then lemma 2 and the chain rule imply that $\exists x_2 \in Int(S)$ such that $\varphi(x_2) \in Int(S)$, φ differentiable at x_2 and $\partial\varphi(x_2) \leq -1$. For $\psi(1) - \psi(0) > 0$ an identical conclusion follows. Step 2: $N > 2$. Suppose $(x_1^a, ..., x_N^a)$ is an asymmetric equilibrium, where we can assume $x_1^a \neq x_2^a$ without loss of generality. Take $X = (x_3^a, ..., x_N^a) \in S^{N-2}$ as an exogenously fixed parameter vector and suppose players $g = 1, 2$ play a two-player game, treating X as fixed. Then (x_1^a, x_2^a) as well as (x_2^a, x_1^a) must be asymmetric equilibria of this symmetric, parametrized two-player game. Thus, by step 1, if the N-player game has an asymmetric equilibrium $\exists\ X \in S^{N-2}$ and $x_2 \in Int(S)$ such that $\partial\varphi(x_2; X) \leq -1$, which completes the proof.

■

7.3 Proof of theorem 3

Let $N = 2$ and suppose (x_1^a, x_2^a) is an asymmetric equilibrium. Then (x_2^a, x_1^a) also is an asymmetric equilibrium and $\varphi(x_2^a) = x_1^a$, $\varphi(x_1^a) = x_2^a$. Define $\psi_i(t_i) \equiv \varphi_i\left(x_1^a + t_i(x_2^a - x_1^a)\right)$, where $i = 1, 2$ and $t_i \in [0, 1]$. Then $\psi_i(0) = \varphi_i(x_1^a)$ and $\psi_i(1) = \varphi_i(x_2^a)$. Note that $\psi_i(0) \neq \psi_i(1)$ for at least one i. Moreover, $\psi_i \in C\left([0, 1], S_i\right)$ and, according to the chain rule, if $\psi_i(t_i) \in Int(S_i)$ the function ψ_i is differentiable expect possibly for a set of isolated points by presupposition. Hence if $\varphi_i(x_1^a + t_i(x_2^a - x_1^a)) \in Int(S_i)$ and φ_i is differentiable at the point $x_1^a + t_i(x_2^a - x_1^a)$, then the chain rule implies:

$$\psi_i{}'(t_i) = \partial\varphi_i\left(x_1^a + t_i\left(x_2^a - x_1^a\right)\right) \cdot \begin{pmatrix} \psi_1(0) - \psi_1(1) \\ \psi_2(0) - \psi_2(1) \end{pmatrix} \tag{15}$$

The proof now is case-by-case. (I) $\psi_i(0) = \psi_i(1)$ for one i. Suppose that $\psi_1(0) = \psi_1(1)$ and hence $\psi_2(0) \neq \psi_2(1)$. Then, similar to step 1 of the proof of theorem 2, lemma 2 and (15) imply that $\exists\ x_2' \in S_1 \times Int(S_2)$ such that $\delta(x_2') \leq -1$ is satisfied. Similarly, if $\psi_2(0) = \psi_2(1)$ then $\alpha(x_2) \leq -1$ for some $x_2 \in Int(S_1) \times S_2$. Consequently, $\alpha(x_2), \delta(x_2) > -1$ for any $x_2 \in S$ where the respective derivative exist, rule out the possibility of asymmetric equilibria with a similar i-th projection, and henceforth assume this condition to be satisfied. Further, suppose that $\psi_i(0) \neq \psi_i(1)$ for $i = 1, 2$ and define $m \equiv \frac{\psi_2(0)-\psi_2(1)}{\psi_1(0)-\psi_1(1)}$. (II) $\psi_i(0) > \psi_i(1)$ or $\psi_i(0) < \psi_i(1)$, $i = 1, 2$, hence $m > 0$. Suppose that $\psi_i(0) > \psi_i(1)$. Then lemma 2 and (15) assert the existence of $x_2, x_2' \in S$ such that $\alpha(x_2) + m\beta(x_2) \leq -1$ and $\gamma(x_2')\frac{1}{m} + \delta(x_2') \leq -1$. Eliminating m gives $\beta(x_2)\gamma(x_2{}') \geq (1 + \alpha(x_2))(1 + \delta(x_2{}'))$. The same conclusion holds if $\psi_i(0) < \psi_i(1)$. (III) $\psi_1(0) < \psi_1(1)$ and $\psi_2(0) > \psi_2(1)$ (or opposite inequalities), hence $m < 0$. Then proceed as in (II) to obtain the same conclusion as in case (II). The above derivation implies that whenever (4) is satisfied, there cannot be any asymmetric equilibria. This proves the claim for $N = 2$, and the proof is completed by the same logic as in step 2 of the proof of theorem 2.

■

Remark I: Theorem 3 can be extended to the case, where $\varphi(x_{-1}) \in \partial S$ is possible under our usual assumptions on Π from section 2. To see why and how let $k = N = 2$, and suppose that $\varphi_2(x_2^0) = \bar{S}_2$ for some $x_2^0 \in S$, but $\varphi_1(x_2^0) \in Int(S_1)$. Now consider the following two systems of equation:

$$I)\quad \Pi_1(\hat{x}_{11}, \bar{S}_2, x_2^0) = 0 \qquad II)\quad \begin{array}{l} \Pi_1(\tilde{x}_{11}, \tilde{x}_{12}, x_2^0) = 0 \\ \Pi_2(\tilde{x}_{11}, \tilde{x}_{12}, x_2^0) = 0 \end{array}$$

As $\varphi_1(x_2^0) \in Int(S_1)$ our assumptions on Π imply that, for fixed $x_{12} = \bar{S}_2$, equation I) implicitly defines a local C^1-function $\hat{\varphi}_1(x_2)$, with $\hat{\varphi}_1(x_2^0) = \varphi_1(x_2^0)$.

The technical difficulty that $\varphi_2(x_2^0) \in \partial S_2$ potentially[18] imposes, is that II) can have a local C^1-solution $(\tilde{x}_{11}, \tilde{x}_{12})$, with $\tilde{x}_{11} = \tilde{\varphi}_1(x_2)$ around x_2^0, but both $\hat{\varphi}_1(x_2) \neq \tilde{\varphi}_1(x_2)$ as well as $\partial\hat{\varphi}_1(x_2) \neq \partial\tilde{\varphi}_1(x_2)$ are possible. If II) has a solution both $\hat{\varphi}_1(x_2), \tilde{\varphi}_1(x_2)$ are local C^1-functions around x_2^0, and $\varphi_1(x_2) = \hat{\varphi}_1(x_2)$ or $\varphi_1(x_2) = \tilde{\varphi}_1(x_2)$ around x_2^0. Together with the previous result, this shows that $\varphi_1(x_2)$ may not be differentiable at or around[19] x_2^0 despite that $\varphi_1(x_2^0) \in Int(S_1)$.

With this insight we can adapt the proof of theorem 3 to obtain a similar condition as (4). To see how, let $x_2 \neq x_2'$ and let $\psi(t) \equiv \varphi(x_2 + t(x_2' - x_2))$, $\hat{\psi}_1(t) \equiv \hat{\varphi}_1(x_2 + t(x_2' - x_2))$ and $\tilde{\psi}_1(t) \equiv \tilde{\varphi}_1(x_2 + t(x_2' - x_2))$ for $t \in [0, 1]$ and assume[20] that $\psi_2(t) = \bar{S}_2$ for some t.

[18] If the point $(\varphi_1(x_2^0), \bar{S}_2)$ is not a solution of II), then $\varphi_1(x_2)$ is implicitly defined by I) as a C^1-function around x_2^0. In sloppy terms this means that the boundary solution $\varphi_2(x_2) = \bar{S}_2$ is "strict", and the following problem does not emerge.

[19] If $k > 1$ and there are boundary solutions non-differentiable points need not be locally isolated.

[20] The following argument can easily be adjusted to capture the case where $\psi_2(t) = 0$ may also occur.

Suppose that $A_0 \equiv \psi_1(0) > \psi_1(1) \equiv A_1$. We want to show that there is $t \in (0,1)$ such that either $\hat{\psi}'_1(t) \leq \psi_1(1) - \psi_1(0)$ or $\tilde{\psi}'_1(t) \leq \psi_1(1) - \psi_1(0)$. By contradiction, assume that $\hat{\psi}'_1(t) > \psi_1(1) - \psi_1(0)$ and $\tilde{\psi}'_1(t) > \psi_1(1) - \psi_1(0)$ whenever these objects exist. Geometrically, this means that the functions $\hat{\psi}_1, \tilde{\psi}_1$ are less step (perhaps even increasing) than the line connecting A_0 and A_1. That is, for any $t_0 \in (0,1)$ there is a perfect interval $\mathbb{B} = (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $\hat{\psi}_1$ or $\tilde{\psi}_1$ are moving away from the line connecting A_0 and A_1 as t increases on $\mathbb{B}$ whenever these functions are well-defined at t_0.

The fact that $\psi_1(t_0)$ corresponds either to $\hat{\psi}_1(t_0)$ or to $\tilde{\psi}_1(t_0)$ whenever $\psi_1(t_0) \in Int(S_1)$ then implies that if $\psi_1(t_0) \in Int(S_1)$, the function $\psi_1(t)$ must always be moving away from the line connecting A_0 with A_1, which, by continuity, makes $\psi_1(1) = A_1$ impossible, contradiction.

The consequence of this argument is that if $\varphi(x_2) \in \partial S$ can occur, we must apply the reasoning in the proof of theorem 3 to the function $\hat{\varphi}_1(x_2)$ as well. In practice this means that we have to determine the slopes in condition (4) not only by applying the IFT to the system II) (this is sufficient if we know that best-replies are always interior) by also by applying the IFT to the FOC with boundary points. For example, applying the IFT to $\Pi_1(x_{11}, x_{12}, x_2) = 0$, where $x_{12} = 0$ or $x_{12} = \bar{S}_2$ are held fixed, gives the slopes

$$\hat{\alpha}_1 = \frac{\partial x_{11}(\bar{S}_2, x_{21}, x_{22})}{\partial x_{21}}, \hat{\beta}_1 = \frac{\partial x_{11}(\bar{S}_2, x_{21}, x_{22})}{\partial x_{22}}, \hat{\alpha}_2 = \frac{\partial x_{11}(0, x_{21}, x_{22})}{\partial x_{21}}, \hat{\beta}_2 = \frac{\partial x_{11}(0, x_{21}, x_{22})}{\partial x_{22}}$$

The same argument applied to Π_2 gives four additional slopes $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\delta}_1, \hat{\delta}_2$. Now, working through the same steps as in the proof of theorem 3 shows that if the statement in (4) *additionally* holds for *any* combination of these new slopes (where we e.g. replace α by $\hat{\alpha}_1$, $\beta = \hat{\beta}_2$...) evaluated at all $x_2, x'_2 \in S$, this is sufficient to rule out the possibility of asymmetric equilibria in the game.

Remark II: Theorem 3 extends to the case $k > 2$. To illustrate this, suppose $k > 2$, $N = 2$ and consider the two asymmetric equilibria (x_1^a, x_2^a) and (x_2^a, x_1^a). For simplicity, assume that $\varphi(S) \subset Int(S)$, i.e. φ is everywhere differentiable. Then $\psi_i(t) \equiv \varphi_i(x_1^a + t(x_2^a - x_1^a))$ is differentiable on $(0,1)$. Let $\Delta_i \equiv \varphi_i(x_1^a) - \varphi_i(x_2^a)$ and $\Delta \equiv (\Delta_1, ..., \Delta_k)$. Then the MVT applied separately to each ψ_i, asserts the existence of k points $x_2^i \in Int(S)$, $1 \leq i \leq k$, such that $\tilde{A} \cdot \Delta = -\Delta$, where $\tilde{A}$ is a $k \times k$ matrix with entries $a_{ij} = \frac{\partial \varphi_i(x_2^i)}{\partial x_{2j}}$, $1 \leq i, j \leq k$. Equivalently, we get that $(I + \tilde{A}) \cdot \Delta = A \cdot \Delta = 0$ where

$$A = \begin{pmatrix} 1 + \frac{\partial \varphi_1}{\partial x_{21}} & \frac{\partial \varphi_1}{\partial x_{22}} & \cdots & \frac{\partial \varphi_1}{\partial x_{2k}} \\ \frac{\partial \varphi_2}{\partial x_{21}} & 1 + \frac{\partial \varphi_2}{\partial x_{22}} & \cdots & \frac{\partial \varphi_2}{\partial x_{2k}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \varphi_k}{\partial x_{21}} & \cdots & \cdots & 1 + \frac{\partial \varphi_k}{\partial x_{2k}} \end{pmatrix}$$

As $\Delta \neq 0$ the equation $A \cdot \Delta = 0$ implies that $Det(A) = 0$. Now suppose that $\Delta_k = 0$. Then

$$A_{k-1} \begin{pmatrix} \Delta_1 \\ \vdots \\ \Delta_{k-1} \end{pmatrix} = 0$$

where A_{k-1} is formed from A by cancelling the k-th row and column. As $(\Delta_1, ..., \Delta_{k-1}) \neq 0$ this equation implies $Det(A_{k-1}) = 0$, where $Det(A_{k-1})$ is a principal minor of order $k-1$ of A. Obviously, if $\Delta_j = 0$ for any $j = 1, ..., k$ then the corresponding principal minor of order $k-1$ of A must be zero. This argument may be continued up to the case that $k-1$ of the k Δ_i's are zero, which implies that at least one principal minor of A must be equal to zero if an asymmetric equilibrium exists. Consequently, if all principal minors of A are non-zero at any points $x_2^1, ...x_2^k \in Int(S)$, we may conclude that there cannot be any asymmetric equilibria.

7.4 Proof of corollary 3

Suppose that there is an asymmetric equilibrium $x^a = (x_1^a, x_2^a, ..., x_N^a)$ with $x_1^a > x_2^a$ but e.g. $\beta \geq 0$ and $\alpha > -1$. Then by case (II) of the the proof of theorem 3 $\exists\ \tilde{x}_2 \in S$ such that $\alpha(\tilde{x}_2) + m\beta(\tilde{x}_2) \leq -1$ for some X. As $m > 0$ this implies that $\beta(\tilde{x}_2) < 0$, a contradiction. Hence there cannot be any strictly ordered equilibria, which proves i), and ii) is proved in the same way. Finally, if $\alpha, \delta > -1$ case (I) of the the proof of theorem 3 shows that there cannot be asymmetric equilibria where two players choose the same component strategies.

■

7.5 Proof of proposition 3

The proof of proposition 3 requires the following lemma.

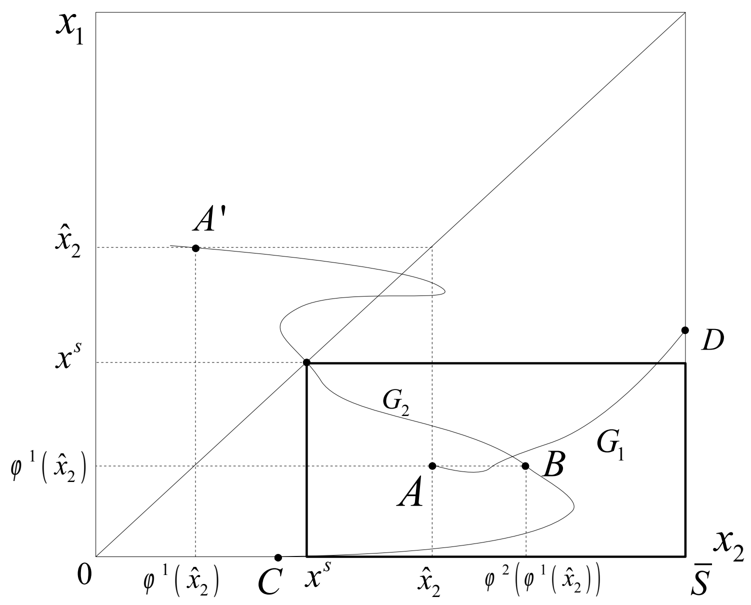
Lemma 3 (Characterization of asymmetric equilibria) *In a symmetric one-dimensional two-player game with $\varphi \in C(S,S)$ no asymmetric equilibria exist if and only if*

$$\varphi\left(\varphi(x)\right) < x \quad \forall x \in S \, \text{with} \, \varphi(x) < x \tag{16}$$

or equivalently

$$\varphi\left(\varphi(x)\right) > x \quad \forall x \in S \, \text{with} \, \varphi(x) > x \tag{16'}$$

Proof: I only prove the claim for (16), the claim for (16') is proved in the same way. "$\Rightarrow$" Suppose that (x_1, x_2) is an asymmetric equilibrium. By symmetry, we can assume that $x_1 < x_2$, i.e. $\varphi(x_2) < x_2$ but $\varphi\left(\varphi(x_2)\right) = x_2$, contradicting (16). "$\Leftarrow$" The proof of this direction naturally is more involved. Let $G_1 \equiv \{(x_1, x_2) \in S^2 : \varphi^1(x_2) = x_1\}$ and $G_2 \equiv \{(x_1, x_2) \in S^2 : \varphi^2(x_1) = x_2\}$ denote the graphs of the best-response functions of the two players. Further, $G_1(x_2) \equiv (\varphi^1(x_2), x_2)$ and $G_2(x_1) \equiv (x_1, \varphi^2(x_1))$ denote specific points on the graphs. The proof is by contraposition. Suppose $\exists\ \hat{x}_2$ such that $\varphi^1(\hat{x}_2) < \hat{x}_2$ but $\varphi^2\left(\varphi^1(\hat{x}_2)\right) \geq \hat{x}_2$. If $\varphi^2\left(\varphi^1(\hat{x}_2)\right) = \hat{x}_2$ then there is nothing to prove as $(\varphi^1(\hat{x}_2), \hat{x}_2)$ obviously is an asymmetric equilibrium, so suppose that $\varphi^2\left(\varphi^1(\hat{x}_2)\right) > \hat{x}_2$. Such a situation is illustrated in figure 7.5 with points $A = G_1(\hat{x}_2) \in G_1$ and $B = G_2\left(\varphi^1(\hat{x}_2)\right) \in G_2$. First, note that $G_2(0) \in \{0\} \times S$, as indicated by the point C. Next, note that, by symmetry, G_2 must pass through a point $A' = G_2(\hat{x}_2)$. By continuity of the best-response function there must be at least one symmetric equilibrium in the interval $(\varphi^1(\hat{x}_2), \varphi^2\left(\varphi^1(\hat{x}_2)\right))$. Let $x^s = min\left\{x_2 : \varphi^1(\hat{x}_2) \leq x_2 \leq \hat{x}_2, \varphi^1(x_2) = x_2\right\}$. Consider the rectangle $[0, x^s] \times \left[x^s, \bar{S}\right]$. By construction, (x^s, x^s) is the only symmetric equilibrium in this rectangle. Moreover, G_2 partitions this rectangle (because G_2 is continuous) and $G_1(\hat{x}_2)$ must lie in the lower partition ("beneath" G_2). But as $G_1(\bar{S}) \in S \times \left\{\bar{S}\right\}$ (indicated with D) and G_1 is continuous there must be an $x^2 \in \left(\hat{x}^2, \bar{S}\right]$ such that $G_1(x_2) \in G_2$. Hence an asymmetric equilibrium



exists.

■

In words, lemma 3 says that if player 1's reaction function lies below the graph of player 2's reaction function and $\varphi^1(x_2) < x_2$, then an asymmetric equilibrium must necessarily exist.

Proof of proposition 3:

By contradiction, suppose that the asymmetric game has an equilibrium with $x_j > x_g$, where $g < j$ (and thus $c_g > c_j$). Consequently, there exists X such that $\varphi^j(x_g; X, c_j) > x_g$ and $\varphi^g(\varphi^j(x_g; X, c_j); X, c_g) = x_g$. As best-replies are increasing on $[\underline{c}, \bar{c}]$ this implies that

$$\varphi^g\left(\varphi^j\left(x_g; X, c_j\right); X, c_g\right) \geq \varphi^g\left(\varphi^j\left(x_g; X, c_j\right); X, c_j\right)$$

Hence there exists x_g such that $\varphi^j(x_g; X, c_j) > x_g$ but $x_g \geq \varphi^g\left(\varphi^j\left(x_g; X, c_j\right); X, c_j\right)$, which in turn by (16') of lemma 3 implies that the symmetric two-player game with best reply function $\varphi(x; X, c_j)$ must have an asymmetric equilibrium, a contradiction. Hence $x_g \geq x_j$, and the result follows by induction. To prove the version for strictly increasing replies, suppose that the asymmetric game has an equilibrium with $x_g = x_j = x$. Thus there exists X such that $\varphi^g(x; X, c_g) = \varphi^j(x; X, c_j) = \varphi^g(x; X, c_j)$, contradicting $\varphi^g(x; X, c_g) > \varphi^g(x; X, c_j)$ as implied by strict monotonicity.

■

7.6 Proof of proposition 4

Lemma 4 *Suppose that* $\phi(\cdot,\cdot) \in C\left(S^N \times \mathcal{P}^N, S^N\right)$ *and consider a symmetric game* $\Gamma(c_0)$*,* $c_0 \in \mathcal{P}^N$*. Suppose that* (x^n) *is a sequence of FPs, i.e.* $\phi\left(x^n, c^n\right) = x^n$*. If* $(x^n, c^n) \to (x_0, c_0)$*, then* x_0 *is an equilibrium of* $\Gamma(c_0)$*.*

Proof: Define $z(x,c) \equiv \phi(x,c) - x$ and note that x is a FP of ϕ iff $z(x,c) = 0$. As $(x^n, c^n) \to (x_0, c_0)$ continuity of z implies $\lim_{n\to\infty} z\left(x^n, c^n\right) = z\left(x_0, c_0\right)$. But $z\left(x^n, c^n\right) = z^n \to 0$ implies that $z(x_0, c_0) = 0$.

■

Proof of proposition 4:

As $x^* \in Int(S^N)$ is regular, $\nabla F(x^*, c) = 0$, and $\nabla F(\cdot,\cdot)$ is continuously differentiable around (x^*, c), the IFT asserts that for any c' in some neighborhood $U \subset \mathcal{P}^N$ of c the equation system $\nabla F(x, c') = 0$ has a locally unique solution $x = h(c')$, where $h \in C^1\left(U, V\right)$ and $V \subset S^N$ is a neighborhood of x^*, which shows existence and local uniqueness of an equilibrium for parameters $c \in U$. Let $E(c)$ denote the set of equilibria of the game with parameter vector c. To see global uniqueness, suppose by contradiction that for every $\delta > 0$ $\exists$ $c^n \in \mathbb{B}(c, \delta)$ such that $E(c^n)$ is multi-valued. Hence there is a sequence (c^n) with $\lim_{n\to\infty} c^n = c$ such that $E(c^n)$ is multi-valued for any $n \in \mathbb{N}$. Consequently, we can find two sequences $(x^n), (y^n)$ with $x^n \neq y^n$ and $\phi(x^n, c^n) = x^n$, $\phi(y^n, c^n) = y^n$ and $x^n \to x^*$. Define $z(x,c) \equiv \phi(x,c) - x$. Because $z(\cdot,\cdot)$ is continuous the set $z^{-1}\left(\{0\}\right) \subset S^N \times \mathcal{P}^N$ is compact. As (y^n, c^n) is a sequence in $z^{-1}\left(\{0\}\right)$ there is a convergent subsequence (y^{n_t}, c^{n_t}), hence also $y^{n_t} \to y^*$. But then lemma 4 and the fact that x^* is unique imply $y^* = x^*$, which by the regularity of x^* means that there is a T such that $y^{n_t} = x^{n_t}$ for all $t \geq T$, a contradiction.

■

7.7 Proof of proposition 8

i) The first claim follows from theorem 1. To see the second claim note that $\breve{J}(x)$ has only negative eigenvalues iff $\breve{J}(x)$ is negative definite, which holds iff (a) $\Pi_{11}(x) < \Pi_{12}(x)$ and (b) $\Pi_{11}(x) + (N-1)\Pi_{12}(x) = \tilde{J}(x_1) < 0$, but (b) holds as x is the only symmetric equilibrium.

ii) Let λ_1, λ_2 be the eigenvalues of $\hat{J}(x_1)$. As $s_1, s_2 > 0$ we must have $\lambda_1+\lambda_2 = Trace(\hat{J}(x_1)) = s_1\tilde{\Pi}_{11} + s_2\tilde{\Pi}_{22} < 0$ as well as $\lambda_1\lambda_2 = Det(\hat{J}(x_1)) = s_1 s_2 Det(\tilde{J}(x_1)) > 0$. Hence λ_1, λ_2 must have negative real parts.

iii) In case of multiple symmetric equilibria theorem 1 implies the existence of $x_1 \in Cr^s$ such that $Det(-\tilde{J}(x_1)) < 0$. As $Det(-\hat{J}(x_1)) = s_1 \cdot \ldots \cdot s_k \cdot Det(-\tilde{J}(x_1))$ also $Det(-\hat{J}(x_1)) < 0$. Consequently, the product of all k eigenvalues of $-\hat{J}(x_1)$ is negative, which further implies the existence of at least one negative eigenvalue. Then, $\hat{J}(x_1)$ has at least one positive eigenvalue, which means that x is symmetrically unstable.

■

7.8 Sufficient conditions for symmetric stability

Proposition 10 *If x is a symmetric equilibrium and $\hat{M}(x_1) \equiv -\frac{1}{2}\left(\hat{J}(x_1) + \hat{J}(x_1)^T\right)$ is positive definite, then the equilibrium is symmetrically stable.*

Proof: $\hat{M}(x_1)$ is positive definite iff $z(-\hat{J}(x_1))z^T > 0$ for $z \neq 0$. But then all eigenvalues of $-\hat{J}(x_1)$ are positive and hence x is symmetrically stable.

■

If e.g. the game is locally supermodular at a symmetric equilibrium ($\Pi^1_{ij}(x) \geq 0$, $1 \leq i \leq k$, $1 \leq j \leq 2k$, $j \neq i$) the following condition might be helpful for establishing symmetric stability:

Proposition 11 *If x is a symmetric equilibrium, $-\tilde{J}(x_1)$ has only non-positive off-diagonal entries and all leading principal minors of $-\tilde{J}(x_1)$ are positive, then x is symmetrically stable.*

Proof: A matrix satisfying all above properties (sometimes called M-Matrix) has only eigenvalues with positive real parts (see e.g. Horn and Johnson (1991)).

■

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