

The relationship between stability and uniqueness in models with sum-aggregative, symmetric or general payoffs

By Andreas Hefti*

June 2014

Abstract

We address the relationship between uniqueness of equilibria and stability with respect to several popular dynamics. It is shown that while in general discrete- and continuous-time stability conditions imply uniqueness, these fundamental properties of an equilibrium set are equivalent mathematical properties of models with symmetric or sum-aggregative payoff functions and non-decreasing equilibrium strategies. The baseline results extends to the case of aggregate-taking behavior, and also to non-strategic models such as Walrasian equilibria. We further use our machinery to reconsider the stability relations of the different dynamics in greater detail, and show that the comparably restrictive nature of discrete dynamics originates in the simultaneity of adjustments in case of symmetric games. Asynchronous choice behavior or heterogeneity in non-myopic forward anticipation may stabilize the adjustment process.

Keywords: Stability; Uniqueness; Noncooperative Games; Aggregate-taking behavior; Symmetric Games; Contraction principle; Dominance solvability; Anticipation

JEL Classification: C62, C72, D43, L13

*Author affiliation: University of Zurich, Bluemlisalpstrasse 10, CH-8006 Zurich. E-mail: andreas.hefti@econ.uzh.ch. Phone: +41 787354964.

1 Introduction

Since Cournot's illustration of a stable equilibrium (Cournot (1897)), i.e. a situation where mutual rational behavior dynamically restores a previous departure of the equilibrium, economists concerned themselves with the stability of an outcome. Stability conditions usually can be stated as requirements on some adjustment matrix (or a function thereof), and as such bear some specific mathematical properties. It is thus natural and important to ask, how these properties are related to other aspects of the model, in particular of the equilibrium set itself.¹

In this context surprisingly few contributions have addressed the *relation* between stability and uniqueness of equilibria, while both aspects have been intensely and repeatedly studied separately in specific examples. This is understandable as uniqueness usually is a global property of a model, but stability frequently is a local concept, at least with non-linear dynamics. This article investigates whether and when stability and uniqueness conditions share a common mathematical structure. Our setting encompasses several different popular dynamics as well as different types of (strategic) behavior. A compact analysis becomes possible as we exploit index theory to obtain a local spectral characterization of when a game has a unique equilibrium. While our exposition is centered on a game-theoretic structure, our results readily extend to (non-strategic) models featuring a comparable mathematical structure, such as e.g. Walrasian price equilibria.

We further use our formal machinery in conjunction with the tractable algebraic structure of games with a sum-aggregative representation of payoffs or symmetric equilibria in symmetric games to study the stability relations of the various dynamics in greater detail.

Main results and related literature We restrict attention to the general class of models compatible with index theory. We first establish that the local stability of several popular dynamics, including the (continuous) gradient or the (continuous and discrete) best-reply dynamics, imply uniqueness of the (Nash) equilibrium. While the converse is not true in general (see Scarf (1960) for an early example), this has been shown to hold for certain specific examples with respect to particular dynamics. For example, Dastidar (2000) shows that uniqueness implies local gradient stability of equilibria under a sign restriction on best-replies in the Cournot game, and Okuguchi and Yamazaki (2008) derive sufficient conditions for

¹As the famous mathematician Henri Poincaré emphasizes: One should be less interested in studying the objects itself, but instead study the relations between them (Poincare, 1905).

when a, by presumption, unique equilibrium in the rent-seeking or the Cournot game are even globally stable under the gradient dynamics. Further, uniqueness in globally supermodular games implies stability with respect to a wide array of different dynamics (Vives (1999)).

We extend the literature by proving that local stability - with respect to all above dynamics - local dominance-solvability and uniqueness are in fact *equivalent* properties for models with a sum-aggregative representation of payoffs and *locally* non-decreasing equilibrium replies. We show that this equivalence also applies to symmetric equilibria of symmetric models as well as to aggregate-taking behavior. Further, we prove that stability of the gradient and continuous-time best-reply dynamics and uniqueness are equivalent properties in models with a sum-aggregative structure and locally decreasing but bounded best replies - a property, which games with homogeneous revenue functions of degree < 1 (such as contests) naturally satisfy.

The symmetric structure of symmetric equilibria allows us to study the stability relations among the different dynamics in greater detail. We demonstrate that the restrictive nature of the discrete dynamics in case of equilibrium substitutes hinges on the simultaneity of adjustments entailed in the definition of this process. If decision-makers take turns or, more importantly, have disharmonized forward conjectures about their opponents' actions, this tends towards stabilizing the dynamics.

Article structure After introducing the notation and baseline assumptions, section 3 discusses the general relation between uniqueness and stability of several (popular) adjustment processes, and also considers the general relation between the various dynamics with respect to their stability. In section 4 we proceed to payoff functions with a sum-aggregative representation, where we also discuss local dominance-solvability, two-player games and games with homogeneous revenue functions as special cases, and consider aggregate-taking behavior as a behavioral variant. Section 5 extends the main insights to symmetric games, clarifies the role of simultaneity versus sequentiality for the resulting stability with the discrete dynamics, discusses the consequences of heterogeneity in forward-anticipations of others' actions, and compares the stability relation of aggregate-taking behavior versus Nash behavior in case of a linear-symmetric game. Finally, in section 6 we briefly comment on the relation between stability, uniqueness and the induced comparative-statics patterns, and illustrate by means of a Walrasian model that our results extend to non-strategic environments. Longer proofs are in section 7. The supplementary material contains the mathematical details from matrix

analysis and differential calculus, as well as a novel, compact treatment of the contraction principle for economics.

2 Primitives and notation

We first introduce the basic notations and concepts.

2.1 The game: Notation and assumptions

Consider a game of $N \geq 2$ players. The joint strategy space is $\mathcal{W} \equiv \mathcal{W}(k) = \prod_{j=1}^N \mathcal{W}^j(k)$, where individual strategy spaces $\mathcal{W}^g(k) \subset \mathbb{R}^k$ are compact and convex. $\text{Int}(\mathcal{W}) \neq \emptyset$ denotes the interior of \mathcal{W} . A point $x_g \equiv (x_{g1}, \dots, x_{gk}) \in S^g(k)$ denotes a strategy of player g . Further, $x_{-g} \in \mathcal{W}_{-g}$ is a particular strategy profile of g 's opponents. Player g 's payoff is represented by $\Pi^g \in C^2(\mathcal{W}, \mathbb{R})$. Next, $\nabla \Pi(x) \equiv (\nabla_j \Pi^j(x))_{j=1}^N$ denotes the Nk -vector obtained by stacking all player gradients (the pseudogradient, Rosen (1965)), and its Jacobian is denoted by $H(x) \equiv \frac{\partial \nabla \Pi(x)}{\partial x}$. Interior (Nash) equilibria require that $\nabla \Pi(x) = 0$. We assume² that each players Hessian $H_g(x) \equiv \frac{\partial^2 \Pi^g(x)}{\partial x_g \partial x_g}$ is negative definite whenever $\nabla \Pi(x) = 0$, and denote the derivative of the joint best-reply $\phi(x) = (\varphi^1(x_{-1}), \dots, \varphi^N(x_{-N}))$ at x by $\partial \phi(x)$. Hence $(N, \mathcal{W}(k), \{\Pi^j\})$ is a twice differentiable k -dimensional N -player game, and henceforth any reference to ‘‘game’’ implicitly invokes the above assumptions. We denote by $A(x)$ the block-diagonal matrix with $H_g(x)$ as block-diagonal entries. Note that under the above assumptions $A(x)$ is negative definite whenever $x \in \text{Int}(\mathcal{W})$ is an equilibrium. Finally, if M is some real $m \times m$ matrix, $\sigma(M)$ is the spectrum of M , i.e. the m -list of all eigenvalues (EV) λ of M , and its spectral radius is $\rho(M) \equiv \max \{|\lambda| : \lambda \in \sigma(M)\}$.

2.2 Dynamics

We now define the various dynamics we want to study. Consider $S \in C^1(\mathbb{R}^{Nk}, \mathbb{R}^{Nk})$ satisfying i) $S(0) = 0$ and ii) $\text{Det } \partial S(0) > 0$. We refer to S as *adjustment cofactor*, and it plays a central role in the different time-continuous adjustment process we shall consider. The assumptions on S are discussed in context of theorem 1.

Continuous-time dynamics These dynamics generally take on the form of a vector field defined over \mathcal{W} . The literature has repeatedly considered adjustment processes defined over

²A standard sufficient condition is that Π^g is strongly quasiconcave in x_g (see Avriel et al. (1981)).

the pseudogradient (e.g. Dixit (1986); Dastidar (2000); Hefti (2013)). We refer to the dynamical system induced by the Nk FOC's, $\dot{x}(t) = S(\nabla\Pi(x(t)))$, and associated Jacobian $\hat{H}(x(t))$, as (generalized) *gradient dynamics*.

Dynamics of the form $\dot{x}(t) = S(\phi(x(t)) - x(t))$, with associated Jacobian $\check{H}(x(t))$, have also been considered (Al Nowaihi and Levine (1985), Vives (1999), Dindos and Mezzetti (2006)). We refer to these as (generalized) *continuous-time best-reply (CTBR) dynamics*.

In applications S usually is given by a constant diagonal matrix with strictly positive diagonal entries (adjustment rates). Our adjustment cofactor is more general as i) it allows the adjustment rates to depend on the state of the system, and ii) the adjustment matrix is not required to be diagonal in case of constant adjustment rates.³

Fact 1 *An equilibrium $x \in \text{Int}(\mathcal{W})$ is locally gradient (locally CTBR) stable if the corresponding Jacobian is a stable matrix at x , i.e. if all EV's have negative real parts.*

Discrete-time dynamics The oldest and probably most intuitive dynamics are the discrete (iterative) dynamics defined over the best-reply function itself (sometimes called ‘‘Cournot dynamics’’): $x^{t+1} = \phi(x^t)$. These dynamics converge locally if ϕ induces a local contraction at $x^* \in \text{Int}(\mathcal{W})$, in which case we call x^* contraction-stable.⁴

Fact 2 *An equilibrium $x \in \text{Int}(\mathcal{W})$ is locally contraction-stable if and only if there is a matrix norm $\|\cdot\|$ such that:*

$$\|\partial\phi(x)\| < 1 \quad \text{or equivalently} \quad \rho(\partial\phi(x)) < 1 \quad (1)$$

In the supplementary material we present, inter alia, a novel, careful and instructive proof of this fact.

Remarks on discrete stability Early work on discrete stability (e.g. Hadar (1966)) was aware of (1) with respect to certain simple norms such as $\|\cdot\|_\infty$ or $\|\cdot\|_1$, which require to check only the magnitude of either the row or column sums of $\partial\phi$. Let $R_m(x) \equiv \sum_{t=1}^{Nk} \left| \frac{\partial\varphi_m}{\partial x_t} \right| < 1$

³One could imagine individual adjustment to depend on the current state of the opponents, e.g. $\dot{x}_j(t) = \sum s_i \nabla_i \Pi^i(x(t))$, a point that is typically emphasized by work on evolutionary game theory. Note that if the (non-diagonal) adjustment matrix S is diagonally dominant with strictly positive diagonal, then $\text{Det}(S) > 0$.

⁴It should be mentioned that if ϕ is non-linear around x^* , then $x^t \rightarrow x^*$, $x^0 \neq x^*$, may hold for some initial values even if $\rho(\partial\phi(x^*)) > 1$. In particular, if ϕ is a diffeomorphism on \mathbb{R}^n , x^* is a regular FP and $\rho(\partial\phi(x^*)) > 1$, then there is a lower-dimensional submanifold (but not a neighborhood) about x^* on which $\phi(x^t) \rightarrow x^*$, provided that at least one EV has $|\lambda| < 1$.

denote the m -th row sum of $\partial\phi(x^*)$, and $C_m(x) \equiv \sum_{t=1}^{Nk} |\frac{\partial\varphi_t}{\partial x_m}| < 1$ denotes the m -th column sum of $\partial\phi(x^*)$. Then it follows from (1) that if either $R_m(x) < 1$ or $C_m(x) < 1$ holds $\forall m = 1, \dots, Nk$ then x is contraction-stable. Another sufficient condition for contraction-stability the early literature typically made use of is diagonal dominance of $H(x)$. While diagonal dominance and $\|\partial\phi(x)\|_\infty < 1$ (the Hadar-condition) are in fact *equivalent* for $k = 1$, this does not generalize to $k > 1$, as diagonal dominance is stronger than the Hadar-condition.⁵

Finally, if $h(\cdot)$ is continuous at x , and x is locally stable, it follows that also $h(x_t) \rightarrow h(x)$. Conversely, while $h(x^t) \rightarrow h(x)$ generally does not assert $x^t \rightarrow x$ (consider $h(x) = x$), this may likely hold for certain specific functions, such as aggregates or averages.⁶

3 Stability and uniqueness: The general case

We now connect the local stability of the above dynamics to uniqueness, and compare the dynamics with respect to their local stability. The main point is that stability under any of the above dynamics is a sufficient condition for uniqueness.

A spectral representation of the index theorem The index theorem, the most general approach towards uniqueness in “regular” models (Furth (1986), Mas-Colell et al. (1995), Vives (1999)), asserts that e.g. a “regular” game has a unique equilibrium if and only if the zeros of the vector field $\nabla\Pi$ have index +1, i.e. $Det(-H(x)) > 0$ on the set of critical points $Cr \equiv \{x \in Int(\mathcal{W}) : \nabla\Pi(x) = 0\}$. We refer to games where $\nabla\Pi(x)$ points into $Int(\mathcal{W})$ along the boundary⁷ of \mathcal{W} and $Det(H(x)) \neq 0$ on Cr as *index games*. The analytical power of index theory stems from the fact that it provides us with a local condition to verify uniqueness, which makes it attractive to applied and theoretical researchers likewise. In applications, especially if solving $\nabla\Pi(x) = 0$ explicitly is not possible, we can thus use the mathematical structure implied by $\nabla\Pi(x) = 0$ to evaluate the sign of $Det(-H(x))$.

We provide an answer to the main question of this article by examining how reconcilable the index condition is with local stability conditions. It is therefore convenient to reformulate

⁵See section 8.2 in the supplementary material.

⁶In the supplementary material (section 8.3) it is shown that indeed convergence of the average implies convergence of best-replies almost surely in case of a linear, symmetric game.

⁷The boundary condition implies that $\phi(\mathcal{W}) \subset Int(\mathcal{W})$. Conversely, if payoffs are continuous and strongly quasiconcave in own strategies it follows that, for $k = 1$, the boundary condition holds if $\phi(\mathcal{W}) \subset Int(\mathcal{W})$.

the index condition in terms of the local spectrum of $\partial\phi$:

Proposition 1 *Any index game satisfies $\text{Re}(\lambda_m(x)) \neq 1$ or $\text{Im}(\lambda_m(x)) \neq 0$ for any $\lambda_m(x) \in \sigma(\partial\phi(x))$, $\forall x \in Cr$, and has a unique equilibrium if and only if:*

$$\prod_m (1 - \lambda_m(x)) > 0 \quad , x \in Cr \quad (2)$$

We now present our first result.

Definition 1 *$x \in Cr$ is called potentially stable if x is contraction-stable or if there is some adjustment cofactor $S(\cdot)$ (not necessarily constant nor diagonal) such that x is either gradient or CTBR stable.*

Theorem 1 *Suppose that in a k -dimensional index game any $x \in Cr$ is potentially stable. Then a unique equilibrium exists.*

Theorem 1 thus implies that the stability under any of the standard dynamics discussed above asserts uniqueness. Moreover, we see that if an index game has multiple equilibria, then there must be at least one completely unstable equilibrium x in the sense that we cannot find a neighborhood around x and a dynamical system as introduced above such that this system converges locally to x .

In a nutshell, uniqueness is more general than stability because the index at a zero pertains only to a specific aspect of a mapping's local geometry, namely whether the mapping is locally orientation-preserving, which is less restrictive than the conditions on the eigenvalues of the associated Jacobian required by stability.

We now present an intuition for theorem 1 for the gradient dynamics⁸ in view of the assumptions imposed on the cofactor S . While the first assumption should be clear – if $S(0) \neq 0$ critical points are not rest points of the dynamical system⁹ – the second needs to be commented on. Let x be a zero of $\nabla\Pi$. Assumption ii) geometrically means that the linear transformation $\partial S(0)$ locally preserves the orientation induced by the linear transformation $H(x)$. If this condition is violated, the imputed direction of the adjustments may be unintuitive.¹⁰ Theorem 1 says that a potentially stable equilibrium x must always correspond to a

⁸A similar intuition applies to the CTBR dynamics.

⁹If S is not a constant matrix, the dynamics may have more rest points than there are equilibria, a fact that frequently occurs, e.g., with the replicator dynamics in evolutionary game theory.

¹⁰This can be seen most clearly in case of a constant diagonal matrix. Then $\text{Det}(S) < 0$ iff there is an odd number of negative diagonal entries, but $s_j < 0$ implies the corresponding strategy x_j to increase if $\nabla_j \Pi^j(x) < 0$, contradicting the direction suggested by optimality.

zero of $\nabla\Pi(x)$ with an index of $+1$. This conclusion is valid, in general, only if the cofactor does not change the index of $\nabla\Pi(x)$, which is precisely what assumption ii) asserts.

Comparing the stability relations In complete generality, there is no well-ordered relation between the different stability types, but by either restricting S or the underlying game some order relations emerge. We concentrate only on those adjustment cofactors S , where the projections depend only on the arguments with the same coordinate, i.e. $S(y_1, \dots, y_m) = (s_1(y_1), \dots, s_m(y_m))$. We call such S a *diagonal cofactor*, and a *homogeneous* diagonal cofactor if additionally $s_j(\cdot) = s(\cdot)$ for any player j . As before, the standard case of constant adjustment rates is included. It turns out that CTBR and gradient stability are the same formal properties of one-dimensional models with non-decreasing equilibrium replies.¹¹

Proposition 2 *Suppose that $H(x^*)$ has only non-negative off-diagonal entries and S is an arbitrary diagonal cofactor. For $k = 1$ the following statements are equivalent: i) $H(x^*)$ is stable, ii) $I - \partial\phi(x^*)$ is stable, iii) $\hat{H}(x^*)$ is stable, iv) $\check{H}(x^*)$ is stable. Moreover, x^* contraction-stable is sufficient for ii) if $k = 1$. Finally, i) and iii) are equivalent also for $k > 1$.*

Note that as $-\hat{H} = \partial\hat{S}(0)(-A)(I - \partial\phi)$ and $-\check{H} = \partial\check{S}(0)(I - \partial\phi)$ the CTBR and gradient dynamics are not generally equivalent (see section 7.2). Moreover, if $k = 1$ and x^* is contraction-stable, x^* is always CTBR and gradient stable as well (for diagonal cofactors).

A practical consequence of proposition 2 is that to verify stability with respect to some continuous-time dynamics, it suffices to analyze $H(x^*)$ or $I - \partial\phi(x^*)$, which usually is simpler.

In case of a *homogeneous* diagonal cofactor the following relation between CTBR and contraction-stability holds (for $k \geq 1$):

Proposition 3 *If $x^* \in \text{Int}(\mathcal{W})$ is a contraction-stable equilibrium and S is a homogeneous diagonal cofactor, then x^* is CTBR stable.*

Proof: $\check{H}(x^*) = s(0)(-I + \partial\phi(x^*))$, so $\lambda \in \sigma(\partial\phi(x^*)) \Leftrightarrow s(0)(\lambda - 1) \in \sigma(\check{H}(x^*))$. Hence $\rho(\partial\phi(x^*)) < 1$ implies that every EV of $\check{H}(x^*)$ has negative real part. ■

Notably, homogeneity is generally indispensable for proposition 3; with heterogeneity x^* can be contraction-stable but not CTBR stable (section 7.2, example 2). Moreover, proposition 3

¹¹Such a result does not hold for local substitutes, counterexamples can be easily constructed.

is limited to the CTBR dynamics and is *not* satisfied, in general, for the gradient dynamics, as an equilibrium can be contraction-stable but not gradient stable (even if $S = I$, see section 7.2, example 1), contrary to what is sometimes alleged by the literature.

4 Sum-aggregative payoffs

Economists and applied game theorists frequently study models, where the actions taken by other players affect an agent's payoff only in some aggregate manner. In some cases this aggregate quantity corresponds to the sum of all actions, and payoffs take on the form $\Pi^j = \Pi^j(x_j, Q)$, $Q \equiv \sum x_i$. The most intensively studied example is the Cournot model of quantity competition: $\Pi^j = P(Q)x_j - c_j(x_j)$, but also rent-seeking games, or (Tullock) contests, with payoffs $\Pi^j = \pi\left(\frac{x_j}{Q+r_j}\right) - c_j(x_j)$ have received much attention (Konrad (2009)). We consider one-dimensional games, where on C^r the slope matrix $\partial\phi(x)$ is of the form

$$M = \begin{pmatrix} a_1 & b_1 & \cdots & \cdots & b_1 \\ b_2 & a_2 & b_2 & \cdots & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_N & b_N & \cdots & \cdots & a_N \end{pmatrix} \quad (3)$$

where $a_j = 0$ and $b_j = -\frac{\Pi_{12}^j + \Pi_{22}^j}{\Pi_{11}^j + 2\Pi_{12}^j + \Pi_{22}^j}$. Note that purely sum-aggregative payoffs as in the Cournot or contest example are only a subclass of the games that imply a slope matrix of the form (3). In particular the following theorems are valid for games that have an isomorphic representation as a sum-aggregative game, in the sense that there exists a homeomorphism on the strategies such that the transformed payoff-function is sum-aggregative and preserves the strategic behavior. For example, if payoffs are of the form $\Pi^j(x_j, \sum f^i(x_i))$, where f^i e.g. is a strictly increasing C^1 -function $\forall i$, then setting $e_i = f^i(x_i)$ gives a sum-aggregative representation of the game with similar strategic behavior.¹² In particular our results also apply to the cases, where some weighted average of the joint actions matters, i.e. where payoffs have the form $\Pi^j(x_j, \sum \alpha_i x_i)$.

4.1 Stability and uniqueness

Payoffs with a sum-aggregative structure have an algebraically special structure, which significantly tightens the uniqueness-stability relation. Moreover, there is a close connection to

¹²If $e(t) \rightarrow e^*$, then, by continuity, also $x(t) \rightarrow x^*$.

local dominance solvability.

Local dominance solvability A game is locally dominance solvable at x^* if there exists a neighborhood $V(x^*)$ such that the individual successive elimination of dominated strategies shrinks the joint strategy to the singleton $\{x^*\}$ on $V(x^*)$. Let $\partial\phi(x^*)^+$ denote the matrix derived from $\partial\phi(x^*)$ by replacing each entry with its absolute value. For a one-dimensional game with $\rho(\partial\phi(x^*)^+) \neq 1$ we have that x^* is dominance solvable if and only if¹³ $\rho(\partial\phi(x^*)^+) < 1$ (Moulin (1984)). As $\rho(\partial\phi(x)) \leq \rho(\partial\phi(x)^+)$ contraction-stability is less restrictive than dominance solvability but in case of equilibrium complements or substitutes (i.e. all non-zero entries of $\partial\phi(x^*)$ have the same sign) the two properties are the same.

We now state the main result of this section:

Theorem 2 (Stability and uniqueness) *Let S be an arbitrary diagonal cofactor, and consider an index game, where $\partial\phi(x)$ is given by (3) whenever $x \in Cr$. If $b_j(x) \geq 0 \forall j$ on Cr then the following statements are equivalent: i) there is a unique equilibrium, ii) every $x \in Cr$ is contraction-stable, iii) every $x \in Cr$ is gradient stable, iv) every $x \in Cr$ is CTBR stable, v) every $x \in Cr$ is dominance solvable. If $b_j(x) \in (-1, 0] \forall j$ on Cr , then i), iii) and iv) are equivalent.*

Hence if i) $b_j(x) > -1$ on Cr and ii) all $b_j(x)$ have the same sign whenever $x \in Cr$, then uniqueness, gradient and CTBR stability are the same formal properties, and if especially strategies are local complements the equivalence extends to contraction-stability and dominance solvability. Thus genuinely different concepts such as the index, referring to a topological-algebraic property of a vector field, and contraction, a metric-analytical property of a function, are the same formal properties of a game with local complements.¹⁴

Theorem 2 generalizes a result of Dastidar (2000) (proposition 2), who shows that uniqueness of equilibrium in the Cournot model implies gradient stability if all b_j have the same sign.¹⁵

¹³Knife-edge cases can be constructed, where x^* is dominance solvable but $\rho(\partial\phi(x^*)^+) = 1$. But then i) $\partial\phi(x^*)$ could be singular, meaning that it cannot be an index game, and ii) this knife-edge case is not robust to small perturbations. Therefore, we rule out this case by assumption.

¹⁴The equivalence between uniqueness and stability hinges on the special structure of $\partial\phi$ in sum-aggregative games. Generally, the uniqueness-condition ($\text{Det}(-H(x)) > 0$) needs not imply stability with local complements (see example 6, section 7.2).

¹⁵At this point we would like to mention a mistake in Dastidar (2000) (p. 210), where the author wrongly claims that (gradient) stability of $H(x)$ of a sum-aggregative game also implies D -stability of $H(x)$. While this is the case if $N = 2$ or the off-diagonal entries all have the same sign and $b_j(x) > -1$, it does *not* hold, in general, if the b_j have different signs (see example 4, section 7.2).

Note that the premise of theorem 2 restricts the best-reply slopes only on Cr . Individual replies may have different signs outside of Cr and could be non-monotonic. In particular, the game needs not be super- nor submodular.

With locally decreasing best-replies (local strategic substitutes) contraction-stability is generally more restrictive compared to the other dynamics, but not if $N = 2$ (see proposition 4 below). This difference to complements occurs because contraction-stability imposes a restriction on the collective response (in absolute values), whereas uniqueness requires this condition to hold only with complements, but *not* with substitutes.

Homogeneous revenue functions The equivalence between uniqueness and CTBR or gradient stability holds for several important examples of sum-aggregative games. The main reason is that the requirement $b_j > -1$ naturally holds in many relevant games, not just the Cournot model (where it is equivalent to $P' < c''$). For a general payoff function of the form $\Pi^j(x_j, Q)$ we have $b_j > -1$ iff $\Pi_{11}^j + \Pi_{12}^j < 0$. This tends to be the case in games with homogeneous revenue functions, such as (Tullock) fixed-prize contests:¹⁶

Corollary 1 *Let $\mathcal{W} = \prod_j [\underline{\alpha}_j, \bar{\alpha}_j]$, $\underline{\alpha}_j \geq 0$, and consider a sum-aggregative index game with $\Pi^j = \pi^j(x_j, Q) - c^j(x_j)$, where $c_j' > 0$, $c_j'' \geq 0$ for $x_j > \underline{\alpha}_j$, $\pi_1^j > 0$, $\pi_{11}^j \leq 0$ and $\pi^j(x_j, Q)$ is homogeneous of degree $r < 1$ in (x_j, Q) for $x_j > \underline{\alpha}_j$. Suppose that all $b_j(x)$ have the same sign whenever $x \in Cr$. Such a game has a unique equilibrium if and only if every $x \in Cr$ is gradient or equivalently CTBR stable.*

Proof: By homogeneity $\pi^j(x_j, Q) = Q^r \pi^j(\frac{x_j}{Q}, 1) = Q^r \tilde{\pi}^j(\frac{x_j}{Q})$, and $b_j > -1$ follows as $\Pi_{11}^j + \Pi_{12}^j < 0$ iff $\partial^2 \tilde{\pi}^j(\frac{x_j}{Q})(1 - \frac{x_j}{Q}) + (r - 1)\partial \tilde{\pi}^j(\frac{x_j}{Q}) - \frac{c_j''}{Q^{r-2}} < 0$. The claim follows from theorem 2. ■

It is well-known that best-replies are non-monotonic in the Tullock contest with sum-aggregative representation $\Pi^j = \frac{x_j}{\sum x_i} V_j - c^j(x_j)$. It can be verified that if $n > 2$ (otherwise proposition 4 below applies) and $(V^j, c^j(\cdot))$ are not too different among players, the slopes $b_j(x)$ have the same sign on Cr , and thus uniqueness and stability are equivalent properties in this example.

Two-player games As two-player games have the same formal structure as sum-aggregative games we get the following result (almost) for free:

¹⁶Note that while e.g. the classic Tullock contest is never an index game as $\underline{\alpha}_j = 0$, this can be changed by setting $\underline{\alpha}_j = \varepsilon$, $\forall j$, where $\varepsilon > 0$, but arbitrarily small.

Proposition 4 Consider a two-player index game. If $b_1(x), b_2(x)$ have the same sign whenever $x \in Cr$ then statements *i) - v)* of theorem 2 are equivalent. If $b_1(x), b_2(x)$ have different signs, then statements *i), iii) and iv)* are equivalent.

The difference between two-player games and games with sum-aggregative payoffs is that in the two-player case the grand equivalence result requires only that the slopes have the same sign (possibly negative) at critical points. Further, proposition 4 generalizes a result in Dastidar (2000) (proposition 1) as we show *equivalence* between uniqueness and several stability types, and proposition 4 applies even if a player has $b_j(x) \leq -1$.

4.2 Aggregate-taking behavior

In this section we show that our baseline results on the uniqueness-stability relations extend to the case of aggregate-taking behavior (ATB), which has become prominent in economics and game theory recently (e.g. Alos-Ferrer and Ania (2005)). ATB means that the players best-respond to some aggregate (or average) strategy rather than to individual strategies. A conceptual justification for ATB are perceptual or informational limitations of the players. A further important advantage of ATB is that the algebra frequently becomes much more tractable compared to Nash-behavior (NB), especially in presence of heterogeneity (see e.g. Grossmann et al. (2012)).

ATB: Dynamics The difference to NB, relevant also for the underlying dynamics, is that with ATB the players ignore own effects on the aggregate when deciding.¹⁷ Let $Q = \sum_i x_i$ and $\bar{Q}_j = \sum_{i \neq j} x_i$. With NB the players optimize $\Pi^j(x_j, x_j + \bar{Q}_j)$ for given \bar{Q}_j , whereas with ATB they optimize $\Pi^j(x_j, Q)$ for given Q . Suppose that $\Pi^j(x_j, Q)$ is strongly quasiconcave in x_j , and assume that $\varphi_A^j(Q) \in \text{Int}(S^j) \forall Q$, such that the ATB joint best-reply is given by $\phi_A(x) = \left(\varphi_A^j(Q)\right)_{j=1}^N$, with slopes $\partial\varphi_A^j(Q) \equiv b_A^j(Q) = -\frac{\Pi_{12}^j(x_j, Q)}{\Pi_{11}^j(x_j, Q)}$. An ATB equilibrium x^* solves $\phi_A(x^*) = \phi_A(\sum x_i^*) = x^*$. The ATB gradient and CTBR dynamics are similar to the dynamics of section 3, where we now use the ATB instead of the NB counterparts. The ATB best-reply dynamics are described by the recurrence relation $x^t = \phi_A(x^{t-1}) = \left(\varphi_A^j(\sum x_i^{t-1})\right)_{j=1}^N$. We can use the same machinery as before to establish the following stability properties of an ATB-equilibrium x_A^* :

¹⁷This points towards a Walrasian type of motivation of ATB as the appropriate behavioral assumption if own effects on the aggregate are negligible, as may be the case in large games.

Proposition 5 *Suppose that x_A^* is an interior ATB equilibrium with aggregate Q^* . i) x_A^* is ATB contraction-stable if and only if $\rho(\partial\phi_A(x_A^*)) = |\sum_j b_A^j(Q^*)| < 1$. ii) If S is an arbitrary homogeneous diagonal cofactor, then x_A^* is ATB CTBR stable if and only if $\sum_j b_A^j(Q^*) < 1$. iii) If x_A^* is ATB gradient stable (for some diagonal S) then x_A^* also is ATB CTBR stable for any homogeneous diagonal cofactor.*

If $|\sum_{j=1}^N b_A^j(Q)| < 1$ holds for any $Q(x) = \sum x_i$ with $x \in \text{Int}(\mathcal{W})$, the game has a unique and globally contraction-stable ATB equilibrium. As before, we cannot infer gradient stability from CTBR stability (for arbitrary diagonal S), and vice-versa.¹⁸

Stability and uniqueness Turning to uniqueness we now show that the relationship between uniqueness and stability encountered in theorems 1 and 2 logically extends to ATB. Let $\nabla\Pi_A(x) \equiv \left(\Pi_1^j(x_j, \sum x_i)\right)_{j=1}^N$ be the N -vector obtained by stacking all ATB FOC's, with corresponding Jacobian $H_A(x) = \frac{\partial\nabla\Pi_A(x)}{\partial x}$. Any candidate for an interior ATB equilibrium belongs to $Cr_A \equiv \{x \in \text{Int}(\mathcal{W}) : \nabla\Pi_A(x) = 0\}$. We call a game with ATB, $\phi_A(\mathcal{W}) \subset \text{Int}(\mathcal{W})$ and $\text{Det}(H_A(x)) \neq 0$ on Cr_A an ATB index game. Further, the statement that x is potentially ATB stable has the same meaning as in definition 1.

Theorem 3 (ATB stability and uniqueness) *The following relations between uniqueness and stability are satisfied in an ATB index game. a) If any $x \in Cr_A$ is potentially ATB stable, then there is a unique ATB equilibrium. b) The ATB equilibrium is unique if and only if any $x \in Cr_A$ is CTBR stable for any homogeneous diagonal cofactor. c) If $\sum_j b_A^j(Q(x)) \geq 0$ on Cr_A then the equilibrium is unique if and only if each $x \in Cr_A$ is contraction-stable. d) If $b_A^j(Q(x)) \geq 0$ on Cr_A then the ATB analog of statements i) - v) in theorem 2 are equivalent. e) If $b_A^j(Q(x)) \leq 0$ on Cr_A then there exists a unique ATB equilibrium, which is both gradient and ATB stable (for arbitrary diagonal S).*

Hence the fundamental relations between stability and uniqueness with NB also are satisfied under ATB. Moreover, the equivalence between uniqueness and stability in case of ATB is stronger in such that c) does not necessarily require that all $b_A^j \geq 0$ or e) does not require that $b_A^j > -1$.

¹⁸This is the case essentially because the stability of $(I - \partial\phi_A)$ does not, in general, imply D -stability.

5 Symmetric games

Besides payoffs featuring a sum-aggregative structure, many applications study models with symmetric payoff functions. As with sum-aggregative payoffs symmetric equilibria of symmetric games have a special and highly tractable mathematical structure. If $k = 1$ the slope matrix at a symmetric equilibrium is a special case of (3), without requiring that payoffs are sum-aggregative. As a consequence the previously encountered stability-uniqueness relation, particularly theorem 2, extends to symmetric equilibria. The nice algebraic structure implied by symmetry allows us to consider the uniqueness-stability relation also in higher dimensions, and to study the stability properties of symmetric equilibria in greater detail.

5.1 Uniqueness and stability of symmetric equilibria

In a symmetric game all players have the same strategy space and identical payoff functions. As the “generic” equilibrium type of differentiable symmetric games is symmetric rather than asymmetric,¹⁹ we concentrate more on the former of general symmetric games here. Presuming that $x^* \in \text{Int}(\mathcal{W})$ is a symmetric equilibrium of a k -dimensional N -player game, the slope matrix $M \equiv \partial\phi(x^*)$ takes on the form of a partitioned matrix with $N \times N$ partitions:

$$M = \begin{pmatrix} 0 & A & \cdots & A \\ A & 0 & A & A \\ \cdots & \cdots & \cdots & \cdots \\ A & A & \cdots & 0 \end{pmatrix} \quad (4)$$

where 0 and A both are $k \times k$ matrices and $A = \frac{\partial\varphi^1(x^*)}{\partial x_2}$.

Lemma 1 *The spectral radius of M is $\rho(M) = (N - 1)\rho(A)$.*

To characterize symmetric equilibria we can resort to a reduced form of Π by assuming identically behaving opponents, $\Pi(x_1, \bar{x}, \dots, \bar{x})$. Let $\tilde{\varphi}(\bar{x}) \equiv \varphi^1(\bar{x}, \dots, \bar{x})$, $\bar{x} \in S(k)$, and note that $x^* = (x_1^*, \dots, x_1^*)$ is an interior symmetric equilibrium if and only if $\tilde{\varphi}(x_1^*) = x_1^*$, $x_1^* \in \text{Int}(S^1)$.

Stability of symmetric equilibria We now present the various stability conditions in case of symmetric equilibria:

¹⁹See Hefti (2013) for a recent treatment.

Proposition 6 *Suppose that $x^* \in \text{Int}(\mathcal{W})$ is a symmetric equilibrium. If $k \geq 1$ then x^* is contraction-stable if and only if $(N-1)\rho\left(\frac{\partial\varphi^1(x^*)}{\partial x_2}\right) < 1$ or, equivalently, $\rho\left(\frac{\partial\tilde{\varphi}(x_1^*)}{\partial \bar{x}}\right) < 1$. If $k = 1$ then x^* is gradient stable or, equivalently, CTBR stable (both for arbitrary diagonal S) if and only if $\frac{\partial\varphi^1(x^*)}{\partial x_2} \in (-1, \frac{1}{N-1})$ or, equivalently, $\tilde{\varphi}'(x_1^*) \in (-(N-1), 1)$.*

For $k = 1$ dominance solvability and contraction-stability of a symmetric equilibrium x^* are equivalent, but symmetry implies an even stronger result:

Corollary 2 *Suppose that $k = 1$ and $x^* \in \text{Int}(\mathcal{W})$ is a symmetric equilibrium. Then the following statements are equivalent: 1) x^* is contraction-stable, 2) x^* is dominance solvable, 3) $R_m(x^*), C_m(x^*) < 1 \forall m$, 4) $H(x^*)$ has a dominant diagonal.*

Proof: By lemma 1 $\rho(\partial\phi(x^*)) = |\tilde{\varphi}'(x_1^*)|$, and 1) \Leftrightarrow 3) because $R_m(x^*) = C_m(x^*) = |\tilde{\varphi}'(x_1^*)|$. 3) \Leftrightarrow 4) holds because $H(x^*)$ is diagonally dominant iff $|\tilde{\varphi}'(x_1^*)| < 1$. ■

Hence symmetry implies the standard sufficient contraction conditions to be both *necessary and sufficient* for contraction-stability. This equivalence, however, is in general a one-dimensional phenomenon.²⁰

Index theorem for symmetric equilibria We call a symmetric equilibrium unique if the game has exactly one symmetric equilibrium. Let $\check{\nabla}\check{\Pi}(x_1) \equiv \nabla_1\Pi^1(x_1, \dots, x_1)$. Hence $\check{\nabla}\check{\Pi} : S(k) \rightarrow \mathbb{R}^k$ defines a vector field over $S(k)$ with corresponding Jacobian $\check{H}(x_1) \equiv \frac{\partial\check{\nabla}\check{\Pi}(x_1)}{\partial x_1}$. Note that any interior symmetric equilibrium $x^* = (x_1^*, \dots, x_1^*)$ satisfies $x_1^* \in Cr^s \equiv \{x_1 \in \text{Int}(S(k)) : \check{\nabla}\check{\Pi}(x_1) = 0\}$. We call a symmetric game, where $\check{\nabla}\check{\Pi}$ points into $S(k)$ at boundary points²¹ and $\text{Det}(\check{H}(x_1)) \neq 0$ on Cr^s a symmetric index game. Every index game also is a symmetric index game (the converse is generally false), and a symmetric index game has exactly one symmetric equilibrium if and only if $\text{Det}(-\check{H}(x_1)) > 0$ on Cr^s . Note that $x = (x_1, \dots, x_1) \in Cr$ whenever $x_1 \in Cr^s$.

Theorem 4 (Uniqueness and stability) *Let $k \geq 1$ and consider a symmetric index game and an arbitrary diagonal S . a) If any $x_1 \in Cr^s$ is contraction-stable, then there is a unique symmetric equilibrium (for $k \geq 1$). If $k = 1$ then the following holds: b) If $\frac{\partial\varphi^1(x)}{\partial x_2} \geq 0$ on Cr^s then uniqueness of the symmetric equilibrium, stability and dominance solvability*

²⁰For $k > 1$ this equivalence generally breaks down, but the sufficiency part still holds (see corollary 5 in the supplementary material).

²¹For $k = 1$ this is equivalent to saying that the game has no symmetric boundary equilibria.

(statements i) - v) in theorem 2) are equivalent. c) If $\frac{\partial \varphi^1(x)}{\partial x_2} \in (-1, 0)$ on Cr^s then there exists a unique symmetric equilibrium, and it is both gradient and CTBR stable.

Proof: Let $\nabla \tilde{\Pi}(x_1, \bar{x}) \equiv \nabla_1 \Pi^1(x_1, \bar{x}, \dots, \bar{x})$, and $\tilde{B}(x_1, \bar{x}) \equiv \frac{\partial \nabla \tilde{\Pi}(x_1, \bar{x})}{\partial \bar{x}}$. Then $-\check{H}(x_1) = -(A_1(x_1, x_1) + \tilde{B}(x_1, x_1)) = -A_1(x_1, x_1) \left(I - \frac{\partial \tilde{\varphi}(\bar{x})}{\partial \bar{x}} \right)$, where A_1 is player one's Hessian matrix, and $\text{Det}(-\check{H}(x_1)) > 0$ whenever $\rho \left(\frac{\partial \tilde{\varphi}(x_1)}{\partial \bar{x}} \right) < 1$, which proves a). For $k = 1$ there is a unique symmetric equilibrium x^* iff $\tilde{\varphi}'(x_1) < 1$ on Cr^s , and b), c) then follow from proposition 6. ■

While the equivalence between stability and uniqueness of symmetric equilibria is invariant to the dimension of the relevant matrix, i.e. to the number of players, in case of equilibrium complements, it is in general a one-dimensional phenomenon, see section 7.2 for a counterexample (example 6).

Overall uniqueness In section 4 we have seen that $\frac{\partial \varphi^1(x^*)}{\partial x_2} > -1$ naturally is satisfied in important examples of sum-aggregative games with equilibrium substitutes. Hefti (2013) shows that if $\frac{\partial \varphi^1(x)}{\partial x_2} > -1$ holds whenever $x \in \text{Int}(\mathcal{W})$, then the symmetric game cannot have any asymmetric equilibria. For such games it then is a consequence of theorem 4 that overall uniqueness and gradient or CTBR stability are equivalent. Specifically, $\frac{\partial \varphi^1(x)}{\partial x_2} > -1$ is satisfied in the Cournot or the Tullock example, showing that for symmetric versions of these games uniqueness, gradient and CTBR stability always are equivalent.

5.2 Stability under sequentiality and forward-anticipation

As for $k = 1$ stability conditions can be stated as interval conditions on individual slopes (proposition 6), we refer to the interval on which $a \equiv \frac{\partial \varphi^1(x^*)}{\partial x_2}$ implies stability according to some dynamics as the corresponding stability radius. Many interesting cases (such as Cournot competition or the Tullock contest) are likely to feature equilibrium substitutes, and then the number of players tends to be detrimental to contraction-stability, but matters far less for the gradient, CTBR dynamics or the sequential contraction dynamics.

It turns out that the comparably restrictive nature of the discrete dynamics of symmetric equilibria²² with equilibrium substitutes originates from the simultaneity of adjustments entailed in its definition: If players take turns and observe rather than choosing simultane-

²²In complete generality, examples can be constructed, where an equilibrium is stable under sequential adjustments but not under simultaneous adjustments and vice-versa (Moulin (1984) and Moulin (1986)).

ously, the corresponding best-reply dynamics have the *same* stability radius, $(-1, \frac{1}{N-1})$, as the gradient or CTBR dynamics.

Sequential dynamics Suppose that players adjust sequentially rather than simultaneously, i.e. they take turns. Let $x^* \in \text{Int}(\mathcal{W})$ be a symmetric equilibrium. For a given (arbitrary) ordering $\{1, \dots, N\}$ of the players the sequential dynamics is defined by the sequence (x^t) with components evolving according to:

$$x_i^{t+1} = \begin{cases} \varphi^i(x_{-i}^t) & \frac{1+t-i}{N} \in \mathbb{N} \\ x_i^t & \text{else} \end{cases} \quad (5)$$

The symmetric equilibrium x^* is sequentially stable if $x^t \rightarrow x^*$ locally. In order to establish $x^t \rightarrow x^*$ we must show that the subsequence $x^N, x^{2N}, x^{3N}, \dots$ converges to x^* . Let $y^t = x^{tN}$ and consider the sequence $y^t = z(y^{t-1})$, where $z(y^{t-1}) = z_N \circ z_{N-1} \circ \dots \circ z_1(y^{t-1})$ with $z_j(y) = (y_1, \dots, \varphi^j(y_{-j}), \dots, y_N)$. If z induces a local contraction, then $y^t \rightarrow x^*$ locally.

The composite mapping z is a local contraction²³ at x^* if and only if $\rho(Z(x^*)) < 1$, where $Z(x^*) = \partial z(x^*) = \partial z_N(x^*) \cdot \dots \cdot \partial z_1(x^*)$. By symmetry, $\partial z_j(x^*)$ can be represented by the $N \times N$ -matrix $\partial z_j = (z_{lm})$ with

$$z_{lm} = \begin{cases} 1 & l = m \neq j \\ a & l = j \neq m \\ 0 & \text{else} \end{cases}$$

Then, $Z(x^*) = \prod_{j=1}^N \partial z_j(x^*)$ is a $N \times N$ -matrix, where the first column is zero. The non-trivial eigenvalues λ of $Z(x^*)$ can be found by solving $\hat{Z}(x^*)v = \lambda v$, where \hat{Z} is obtained from Z by deleting the first row and column. For example, for $N = 4$ we obtain

$$\hat{Z}(x^*) = \begin{pmatrix} a^2 & a(1+a) & a(1+a) \\ a^2(1+a) & a^2(2+a) & a(1+a)^2 \\ a^2(1+a)^2 & a^2(1+a)(2+a) & a^2(3+a(3+a)) \end{pmatrix} \quad (6)$$

Note that the submatrix obtained by deleting the last row and column of \hat{Z} formally corresponds algebraically²⁴ to a situation if there were $N = 3$ players, and deleting the last two rows and columns leads to the case of $N = 2$ players. If $a \geq 0$ and we use the maximum

²³See theorem 6 in the supplementary material.

²⁴Changing the number of players may change the equilibrium x^* and associated slope a .

row-sum norm to bound the spectral radius, we obtain that $\rho(\hat{Z}) < 1$ if $a < 1/(N - 1)$ for $N = 2, 3, 4$. For $a < 0$ using the same norm reveals that $\rho(\hat{Z}) < 1$ if $a > -1$, for $N = 2, 3, 4$. Explicit calculation of the spectral radius clarifies that this bound is sharp, i.e. $\rho(Z(x^*)) < 1$ if and only if $a \in (-1, \frac{1}{N-1})$. As the left picture of figure 1 reveals, the EV's of \hat{Z} in dependence of (a, N) are contained in $(-1, \frac{1}{N-1})$ also for $N > 4$. Accordingly gradient, CTBR and the sequential dynamics have *exactly* the same stability radius.

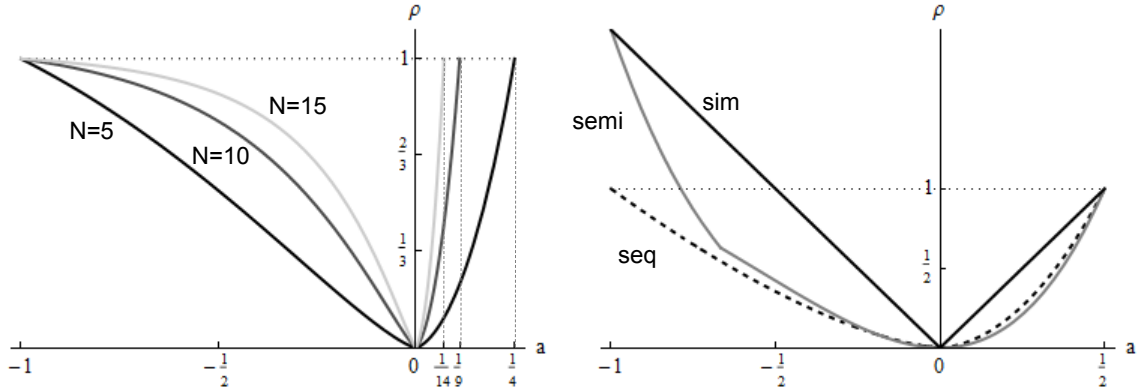


Figure 1: Spectral radius as a function of a

The same result holds for ATB in symmetric games. Suppose that $x_A^* \in \text{Int}(\mathcal{W})$ is a symmetric ATB equilibrium, and $\partial\varphi_A^j(x_A^*) > -1$. It follows from section 4.2 that the stability radius of the (simultaneous) ATB contraction-dynamics of x_A^* is $(-\frac{1}{N}, \frac{1}{N})$, and $(-\infty, \frac{1}{N})$ for CTBR or gradient dynamics.

While the numerics are standard, it is surprisingly hard to formally prove that with NB the stability radius of the sequential dynamics is equivalent to the stability radii of the other time-continuous dynamics, mainly because conventional norms do not efficiently bound the spectral radius. Nevertheless, the claim can be proven for ATB, and we summarize these facts in the next proposition:

Proposition 7 *Suppose that $x^*, x_A^* \in \text{Int}(\mathcal{W})$ are symmetric equilibria under NB, resp. under ATB, and $\partial\varphi_A^j(x_A^*) > -1$. Then the stability radii of the sequential, the gradient and the CTBR dynamics coincide for NB and ATB.*

Finally, the numerics also support the conjecture that the more frequently the players update, i.e. the less sequentially they behave, the smaller the stability radius becomes, gradually approaching the simultaneous stability radius. This is illustrated in the right picture of figure 1 for $N = 3$, where “sim” is the simultaneous and “seq” the sequential dynamics.

”Semi” corresponds to a case, where players 2 and 3 behave sequentially (as in seq) but player 1 updates every period. This finding suggests that heterogeneity in the timing of adjustments, or believe formation, may effectively work in a stabilizing way, which is our next topic.

Heterogeneous forward anticipation Facing a dynamic choice, decision-makers are likely to forward-anticipate the behavior of their opponents. Now, if this forecasting is of a homogeneous nature, the stability properties of the implied best-reply dynamics correspond to those of the standard contraction dynamics (proposition 8 below). This may change substantially, if there is heterogeneity in the degree of forward thinking. We now illustrate that if players decide simultaneously, but forward-anticipate their opponents’ behavior in a non-synchronic way (e.g. they hold heterogeneous beliefs about the updating pattern), this can have a stabilizing effect.

If e.g. player one forward-anticipates the choice of his opponents by $r = 1$ period, his best-reply is to set $x_1^{t+1} = \varphi^1(\varphi^2(x_{-2}^t), \dots, \varphi^N(x_{-N}^t))$. Hence, if all players forward-anticipate the best-replies of their opponents by one period, the induced best-reply process is $x^{t+1} = \phi \circ \phi(x^t) = \phi^2(x^t)$, or generally $x^{t+1} = \phi^{r+1}(x^t)$, where $r = 0$ corresponds to the myopic best-reply adjustment process. For heterogeneous anticipation rates r_1, \dots, r_N we can write the resulting dynamics compactly as

$$x^{t+1} = \begin{pmatrix} \phi_1^{1+r_1}(x^t) \\ \vdots \\ \phi_N^{1+r_N}(x^t) \end{pmatrix} \quad \phi_j^{1+r_j}(x^t) = \text{Pr}(\underbrace{\phi \circ \dots \circ \phi}_{(1+r_j)}(x_t))$$

To consider the stability consequences of heterogeneous forward-anticipation in a simple case²⁵ suppose that player one forward-anticipates responses by one step, $r_1 = 1$, while all other players behave myopically ($r_j = 0, j > 1$). The induced best-response dynamics are

$$x^t = h(x^{t-1}) = \begin{pmatrix} \varphi^1(\varphi^2(x_{-2}^{t-1}), \dots, \varphi^N(x_{-N}^{t-1})) \\ \varphi^2(x_{-2}^{t-1}) \\ \vdots \\ \varphi^N(x_{-N}^{t-1}) \end{pmatrix}$$

²⁵We have examined several different anticipation patterns, both for ATB and NB. Frequently, the heterogeneous anticipation dynamics were strictly more stable in the $a < 0$ region, and, generally, they were never found to be less stable than the myopic dynamics ($r = 0$).

At a symmetric equilibrium, the mapping h implies that

$$\sigma(\partial h(x^*)) = \{0, -a, \dots, -a, a(N-2) + a^2(N-1)\}$$

which implies that the stability radius induced by h is $(-1, \frac{1}{N-1})$ if $N < 7$. Hence a small shift in the behavior of one player leading towards asymmetric, non-myopic anticipation immediately gives the same stability radius as the sequential, gradient or CTBR dynamics. Heterogeneity in anticipation is imperative, as with homogeneous forward-anticipation the same stability radius as with the simultaneous contraction dynamics results. This is intuitive as under homogeneity we just jump forward $r+1$ steps in the myopic sequence per iteration:

Proposition 8 *The composition $\phi^{(r+1)}$ is a local contraction at x^* for any $r \in \mathbb{N}$ if and only if ϕ is a local contraction at x^* . Then the sequence $x^t = \phi^{r+1}(x^{t-1})$ converges locally to x^* .*

5.3 Stability: NB versus ATB

Is a behavior contingent on an exogenous aggregate more likely to induce a stable outcome than a behavior that accounts for the fact that own actions influence the aggregate? While this question is difficult to answer in complete generality, especially because usually $x_A^* \neq x^*$, the ATB and NB dynamics can be compared with respect to their stability in case of a linear symmetric game.

Comparing NB and ATB dynamics Note that the FOC of the NB-problem is linear in (x_j, \bar{Q}_j) if and only if the FOC of the ATB-problem is linear in (x_j, Q) . Hence, without loss of generality, we can work with $\Pi^j = Ax_j + B\tilde{Q}x_j - \frac{1}{2}cx_j^2$, where $\tilde{Q} = Q$ under ATB and $\tilde{Q} = x_j + \bar{Q}_j$ under NB. As we are interested only in stability we choose $A = 0$, and set $c > 0$, $2B < c$ and $S^j = [-1, 1]$. Then $x_A^* = x^* = 0$, $b_j = \frac{B}{c-2B}$ and $b_A^j = \frac{B}{c}$.

Proposition 9 *If strategies are complements ($B \geq 0$) then the equilibrium is ATB contraction-stable whenever it is NB contraction-stable. Under substitutes the opposite direction applies.*

Proof: Let $\rho \equiv \rho(\partial\phi)$ and $\rho_A \equiv \rho(\partial\phi_A)$. From proposition 5 we obtain $\rho_A = \left| \sum b_A^j \right| = \frac{N|B|}{c}$. Further $\rho = \frac{(N-1)|B|}{c-2B}$ follows from lemma 1. If $B \geq 0$, then plugging $\frac{(N-1)B}{c-2B} = \varepsilon$ into ρ_A gives $\frac{N\varepsilon}{(N-1)+2\varepsilon}$. Hence $\rho_A < 1$ if and only if $N\varepsilon < (N-1) + 2\varepsilon$, which holds whenever $\varepsilon < 1$, proving the first claim. The second claim follows as $\rho_A = -\frac{BN}{c} > -\frac{BN}{c-2B} > \rho$. ■

As the equivalence of gradient and CTBR dynamics for ATB in symmetric games can be established in the same way as in proposition 6, and the stability range is $b_A^j \in (-\infty, \frac{1}{n})$, it follows that in case of complements the result from proposition 9 also extends to the gradient and CTBR dynamics.

Intuition Proposition 9 reflects the fundamental difference between ATB and NB. With ATB the players ignore their own effects on the aggregate, which causes two related, but potentially conflictive deviations compared to NB that are relevant for stability analysis. First, the ATB dynamics tend to be less stable than the NB dynamics because under ATB a player's own action triggers an own response in the subsequent period (even if the opponents do not change their strategies). Formally, this manifests itself in the fact that the reply matrix $\partial\phi_A$ may have a non-zero diagonal. If all slopes under ATB and under NB were identical (which is never possible in a linear game unless $B = 0$) this own-response effect would necessarily imply that $\rho_A \geq \rho$.

The second deviation originates from the fact that because an NB player takes into account his own effect on the aggregate, this may either soften or intensify his response to a given strategy profile of his opponents compared to an ATB player. This second effect manifests itself in the slopes of the reply functions. To illustrate, consider the decision of player 1 and suppose that $dx_2 > 0$. Then $d\bar{Q}_1 > 0$, which e.g. under complementarity implies $dx_1 > 0$ for both ATB and NB. But $dx_1 > 0$ also means $dQ > 0$, which is taken into account only by the NB-player and, by complementarity, induces him to increase x_1 even further in the first place. Thus for complements player 1's response towards a change of an opponent's strategy is stronger under NB than under ATB, which explains why the slope of his reply function is comparably steeper with NB. But steeper slopes tend towards destabilizing the dynamics. With strategic substitutes this argument is reversed by the same logic, i.e. the fact that the players take into account their own effects on the aggregate flattens their responses and tends towards stabilizing the dynamics. The presence of the above two effects explains why, in general, we cannot expect to find a clear-cut stability ranking between NB and ATB. Nevertheless, proposition 9 shows that at least with linear reply functions the second effect unambiguously is the dominant one.

6 Discussion and conclusion

This article has addressed the mathematical connection between uniqueness and stability with respect to several popular continuous and discrete adjustment processes. We have studied this relation in abstract and more specific algebraic settings, in the context of important particular examples as well as with respect to different types of behavior. Restricting attention to index games in the context of our question is quite natural. If one is interested whether a model or a (possibly parameterized) game with some specific economic context is likely to generate a unique equilibrium (or rather multiple equilibria), specifying the mathematics (e.g. domain choice) such that an index game results frequently is possible and makes sense by the sheer analytical power of index theory. Index games generally are compatible with highly non-monotonic best-replies (other than, e.g., supermodular games).

In general, contraction-stability, gradient or CTBR stability are sufficient for uniqueness. However, many economically meaningful models impose more algebraic structure on payoffs, and we show that the uniqueness-stability relation is much tighter in many relevant cases. Important examples (e.g. pricing games with imperfect substitutes or investment games) allow for a sum-aggregative payoff representation and feature increasing *equilibrium* (but perhaps not globally increasing) replies, and our results prove that for such models fundamental properties as stability, uniqueness or dominance-solvability are the same formal properties. We show that this equivalence extends to symmetric games (in the sense that there cannot be multiple symmetric equilibria), to two-player games with locally same-signed best-replies, as well as to ATB. With equilibrium substitutes contraction-stability is generally more restrictive than uniqueness, but gradient and CTBR stability and uniqueness are equivalent – a result that applies to many prominent examples such as Cournot competition, imperfectly discriminatory contests or payoffs with homogeneous revenue functions.

Concerning stability, no stability order exists without imposing restrictions on the algebraic structure of the model or the adjustment rates. However, presuming homogeneous adjustment rates, contraction-stability implies CTBR stability, but not necessarily gradient stability, contrary to what has been sometimes alleged by the literature.

A comparison of the ATB and the conventional Nash contraction dynamics in case of a linear symmetric game shows that stability of the Nash dynamics always implies stability of the ATB dynamics in case of complements, and vice-versa in case of substitutes, which is

intuitive given the difference in the nature of the two behaviors.

The simple structure of symmetric models allows us to compare the different dynamics in greater detail. We show that the comparably restrictive nature of contraction-stability in case of locally decreasing replies is driven by the simultaneity entailed in the definition of contraction-stability: If the players update their strategies sequentially, the resulting stability condition corresponds to that of the CTBR or gradient dynamics.

A related important observation is that if the players try to forecast their opponents' best-replies, this tends towards stabilizing the dynamics, provided that there is heterogeneity in forward-anticipation. Generally, this suggests that the level of awareness or attention the players devote to a strategic decision situation can influence whether or not a self-restoring equilibrium dynamic can be expected to prevail: The less synchronized the players' effective behaviors or anticipations are, the more likely the equilibrium is to become stable.

In many different strategic environments, such as (experimental) asset markets or oligopolistic (Cournot) competition, there is clear evidence of considerable heterogeneity in forecasting and expectation-formations (e.g. Hommes (2013)). At the same time experimentalists observe stable behavior more frequently than is suggested by theory (e.g. Huck et al. (2002)), and disharmonized anticipation is a novel explanatory candidate, besides strategic or observational learning, because heterogeneous anticipation seems an intuitive presumption in lab experiments. Moreover, a forecasting-based explanation for stability of the discrete dynamics is different from stochastic stability of the hind-sighted myopic best-reply dynamics due to stochastic choice in aggregative games (Dindos and Mezzetti (2006)).

Forward-anticipation and iterated rationality (Ho et al. (1998)) are somewhat related, as we could view players who try to non-myopically anticipate the moves of their opponents as more sophisticated. In this spirit our results suggest that heterogeneity in the levels of sophistication may have a stabilizing effect on the underlying dynamics.

We conclude this article by a brief comment on stability, uniqueness and comparative static sign patterns indicating the more practical relevance of our results, and by showing that our baseline results on the stability-uniqueness relation is not confined to purely strategic behavior and game theory.

Stability, uniqueness and comparative statics In applications one is frequently interested in the comparative-static predictions of a model. The adequate formal tool to investigate how (small) changes in some parameters affect an equilibrium x is the IFT. If x is a stable equilibrium with respect to some dynamics then, for sufficiently small parameter changes, it is assured that the dynamics will converge to a new, nearby equilibrium, which is one reason for why stability is frequently required. One should realize that stability conditions assert that comparative-static sign patterns are driven by direct rather than strategic effects. Let $x \in \text{Int}(\mathcal{W})$ be an equilibrium and $\nabla\Pi(x; c) = 0$, where c is a parameter vector. By the IFT, the comparative statics of x then are $\frac{\partial x}{\partial c} = -H(x)^{-1} \frac{\partial \nabla\Pi}{\partial c}$, where $-H(x)^{-1} = \frac{\text{adj } H(x)}{\text{Det}(-H(x))}$, and $\text{adj } H$ is the adjoint matrix of H . Note that $H(x)$ quantifies the strategic effects (players' responses to other players strategies), and the sign of $\text{Det}(-H(x))$ (i.e. the index of $\nabla\Pi$) has first-order impact on the sign pattern.²⁶ As the proof of theorem 1 shows, stability conditions assert that $\text{Det}(-H(x)) > 0$, meaning that the sign patterns depends on the direct effects of the exogenous change. These facts essentially say that an “unintuitive” comparative-static prediction, such as an equilibrium quantity increase of a player experiencing a sole upward shock in his cost parameter in a Cournot-duopoly, can never be realized as a stable equilibrium. In fact, an index game with such “strategy-driven” comparative statics cannot possess a unique equilibrium. Vice-versa, if we impose (or verify in applications) that critical points in fact are (potentially) stable, then we simultaneously get economically and mathematically “well-behaved” comparative statics and uniqueness of equilibrium.

The uniqueness-stability relation in Walrasian economies Our basic insights on the uniqueness-stability relation are not confined to game theory, but extend to other economic models featuring a related mathematical structure. For example, Walrasian economies with $L + 1$ commodities (markets) are usually described in terms of an L -valued excess demand function $Z(p) = (Z_1(p), \dots, Z_L(p))$, where any normalized (Walrasian) price equilibrium $p \in \mathbb{R}_+^L$ is characterized by $Z(p) = 0$ under certain standard assumptions on preferences and technology.²⁷ Logically, the function $Z(\cdot)$ plays a similar role as $\nabla\Pi(\cdot)$ did in our previous analysis, with $Cr = \{p > 0 : Z(p) = 0\}$. Adjustment processes can be (and in fact were) introduced in the same way as in section 2.2, although they are perhaps less intuitive in the Walrasian context. For example, the gradient-type of dynamics takes on the form $\dot{p}(t) =$

²⁶Mathematically, this follows as the index corresponds to the local orientation of the linear map $H(x)$.

²⁷See Mas-Colell et al. (1995).

$S(Z(p(t)))$. If the index theorem is applicable to Z (if the economy is regular, see Mas-Colell et al. (1995)), we also obtain proposition 1 in this context, provided that $\partial Z(p)$ has a negative diagonal on Cr (without any sign restriction on off-diagonal elements). This condition plays the same role as negative definiteness of A does in the proof of proposition 1. Note that the condition is intuitive (an increase of the own price should decrease the corresponding excess demand), and significantly weaker than gross substitutability but, by the income effects, it is of course not an universal property. It should be clear that theorem 1 also applies to the Walrasian economy, i.e. stability implies uniqueness. As before, more structure on $Z(\cdot)$ (such as symmetry) will give stronger results.

7 Appendix

7.1 Proofs

Proof of proposition 1 Suppose that $x \in Cr$. From the IFT we obtain $H(x) = A(x)(I - \partial\phi(x))$, and negative definiteness implies that $Det(-H(x)) > 0$ if and only if $Det(I - \partial\phi(x)) > 0$. Now, $\lambda_m \in \sigma(\partial\phi(x)) \Leftrightarrow (1 - \lambda_m) \in \sigma(I - \partial\phi(x))$, and the second claim follows from $Det(I - \partial\phi(x)) = \prod_m (1 - \lambda_m)$. From (2) it follows that any real EV must be unequal 1. If λ is a complex EV with conjugate $\bar{\lambda}$, then $(1 - \lambda)(1 - \bar{\lambda}) = (1 - Re(\lambda))^2 + Im(\lambda)^2$, which is unequal zero iff $Re(\lambda) \neq 1$ or $Im(\lambda)^2 \neq 0$. ■

Proof of theorem 1 Suppose that $x \in Cr$. If x is contraction-stable, i.e. $\rho(\partial\phi(x)) < 1$, then (2) must be satisfied. If $\exists S$ such that x is gradient-stable, then all EV of $(-\hat{H}(x))$ have positive real part, and hence $Det(-\hat{H}(x)) > 0$. But $Det(-\hat{H}(x)) = Det(\partial S(0)) Det(-H(x))$ implies $Det(-H(x)) > 0$. If $\exists S$ such that x is CTBR stable, then $-\check{H}(x) = (\partial S(0))(I - \partial\phi(x))$ has only EV with positive real parts. Hence $Det(I - \partial\phi(x)) > 0$, which by the proof of proposition 1 implies that $Det(-H(x)) > 0$. ■

Proof of proposition 2 By presupposition $-H(x^*)$ has only non-positive off-diagonal entries (Z -matrix). For Z -matrices it is known that $-H(x^*)$ is positive stable, i.e. has only EV's with positive real parts, iff $-H(x^*)$ is D -stable, i.e. if $D(-H(x^*))$ is positive stable for any arbitrary positive diagonal matrix (see Hershkowitz (1992)). Hence i) \Rightarrow iii) follows

from $-\hat{H}(x^*) = \partial S(0)(-H(x^*))$, and the converse follows by noting that $-\hat{H}(x^*)$ also is a Z -matrix. If $k = 1$ the same argument and $-H(x^*) = (-A(x^*))(I - \partial\phi(x^*))$ assert that i) \Leftrightarrow ii), and also ii) \Leftrightarrow iv) since $-\check{H}(x^*) = \partial S(0)(I - \partial\phi(x^*))$. Finally, if x^* contraction-stable, then ii) follows because $(I - \partial\phi(x^*))$ must be positive stable. ■

Proof of theorem 2 The proof of theorem 2 requires the following fundamental lemmata regarding (3):

Lemma 2 Consider the matrix in (3) with $a_j, b_j \in \mathbb{R}$. i) For $a_j \neq b_j$ we have $\text{Det}(M) = \prod_j (a_j - b_j) (\sum_j \frac{b_j}{a_j - b_j} + 1)$, and if $a_j = b_j$ for at least one j , then $\text{Det}(M) = a_j \prod_{i \neq j} (a_i - b_i)$. ii) If $a_j < b_j$ and $b_j \leq 0 \forall j$, then M is D -stable.

Note that lemma 2 implies that an EV $\lambda \neq 0$ of $\partial\phi$ either solves $\sum \frac{b_j}{\lambda + b_j} = 1$ or²⁸ $\lambda = -b_j$.

Proof: If $a_j \neq b_j$ subtract the first column from all other columns, multiply out $(a_j - b_j)$ row-wise, add to the first row all other rows and calculate the determinant of the remaining lower-triangular matrix. If $a_j = b_j$ for more than one j , $\text{Det}(M) = 0$. If $a_j = b_j$ for exactly one j we can assume without loss of generality that $j = 1$, and subtracting the first column from all other columns gives a lower-triangular matrix. This proves i) and ii) is a straightforward consequence of Hosomatsu's lemma (Hosomatsu (1969)). ■

The sum-aggregative structure implies the following useful fact for contraction-stability:

Lemma 3 Let x^* be an interior equilibrium of the sum-aggregative game as described above. If $\sum \frac{|b_j(x^*)|}{1 + |b_j(x^*)|} < 1$, then x^* is both contraction-stable and dominance solvable. Conversely, if all $b_j(x^*)$ have the same sign and x^* is contraction-stable (or equivalently dominance solvable), then $\sum \frac{|b_j(x^*)|}{1 + |b_j(x^*)|} < 1$.

Proof: Because $\rho(\partial\phi) \leq \rho(\partial\phi^+)$ we need only consider the case of complements. But if $b_j(x^*) \geq 0$ for all j , $\rho \equiv \rho(\partial\phi(x^*)) \geq 0$ is in fact an eigenvalue of $\partial\phi(x^*)$ according to the Perron-Frobenius theorem, and obviously $\rho \neq -b_j(x^*)$. By lemma 2 we see that if $\sum \frac{b_j}{1 + b_j} < 1$ then $\sum \frac{b_j}{\rho + b_j} = 1$ implies that $\rho(\partial\phi(x^*)) < 1$, which proves the first claim, and the second claim is true because if all b_j have the same sign the spectral radius ρ solves $\sum_j \frac{|b_j|}{\rho + |b_j|} = 1$,

²⁸The second case can occur only if $b_j = b_g$ for at least two players.

which together with $\rho < 1$ implies that $\sum_j \frac{b_j}{1+b_j} < 1$. ■

Condition of lemma 3 can be shown to be more general than the standard norm-based contraction conditions, and the second claim of lemma 3 is false if the b_j do not have the same sign.²⁹ If the inequality in lemma 3 is satisfied for any $x \in \text{Int}(\mathcal{W})$, then the game has a unique and globally contraction-stable equilibrium³⁰ x^* , which of course is very restrictive and rarely met by examples. We turn to the **proof of theorem 2**:

ii) \Leftrightarrow v) is clear. By theorem 1 we only need to prove that uniqueness implies the various stability types. $-H = -A(I - \partial\phi)$ implies that $\text{Det}(-H(x)) > 0$ iff $\text{Det}(I - \partial\phi(x)) > 0$. By lemma 2, $b_j(x) \geq 0$ implies $\text{Det}(I - \partial\phi(x)) > 0$ on Cr iff $\sum \frac{b_j(x)}{1+b_j(x)} < 1$, and i) \Rightarrow ii) follows from lemma 3. Proposition 2 asserts that ii) \Rightarrow iii), iv). If $b_j(x) \in (-1, 0] \forall j$ then, $\text{Det}(I - \partial\phi(x)) > 0$ on Cr iff $\sum \frac{b_j(x)}{1+b_j(x)} < 1$. Consequently, there is a unique equilibrium x^* , and it follows from lemma 2 ii) that $H(x^*)$ is D -stable, which by the proof of proposition 2 implies that x^* is both gradient and CTBR stable. ■

Proof of proposition 4 The "if"-part follows from theorem 1. Let $x \in Cr$ and consider the matrix $M(x) = I - \partial\phi(x)$. Then the matrix $DM(x)$, where D is a diagonal matrix with entries $d_j > 0$, has only EV's with positive real part iff $\text{Det}(M) = 1 - b_1 b_2 > 0$ by the trace-determinant criterion. Therefore $M(x)$ is stable iff $M(x)$ is D -stable, but from $H(x) = A(x)(I - \partial\phi(x))$ it follows that $M(x)$ is stable iff the equilibrium is unique, which - by D -stability - shows the equivalence of i), iii) and iv) for arbitrary sign patterns of b_1, b_2 . The equivalence of i), ii) and v) in case of local complements follows from theorem 2, so we only need to show that uniqueness implies contraction-stability if $b_1(x), b_2(x) \leq 0$, which - by lemma 3 - indeed is the case as $\sum \frac{|b_j(x)|}{1+|b_j(x)|} < 1$ iff $|b_1(x)| |b_2(x)| < 1$. ■

Proof of proposition 5 i) can be proved by using theorem 6 (supplementary material) and lemma 2, noting that the j -th row of the matrix $\partial\phi_A(x)$ has the form (b_A^j, \dots, b_A^j) , i.e. an EV λ of $\partial\phi_A(x)$ either is zero or solves $\lambda = \sum b_A^j$. Alternatively, we can sum up the equations $\varphi_A^j(Q) = x_j$ to obtain the single equilibrium equation $\sum \varphi_A^i(Q) = Q$. Hence x_A^* is contraction-stable if and only if $1 > \rho(\partial\phi_A(x^*)) = \left| \frac{\partial \sum \varphi_A^i(Q^*)}{\partial Q} \right| = \left| \sum_{j=1}^N b_A^j(Q^*) \right|$. Turning

²⁹E.g. $N = 3$ with $b_1 = b_2 = -3/4$ and $b_3 = 1/2$

³⁰Follows from corollary 3 (supplementary material) and the Banach FP theorem.

to ii), consider $-\check{H}_A = s(0)(I - \partial\phi_A)$. Suppose that $\sum b_A^j < 1$ but $(I - \partial\phi_A)$ has an EV λ with negative real part. Lemma 2 then implies that $\lambda = 1 - \sum b_A^j > 0$, a contradiction. Conversely, suppose that $(I - \partial\phi_A)$ only has EV with positive real parts. If all EV's are 1, then $1 = \text{Det}(I - \partial\phi_A) = 1 - \sum b_A^j$ by lemma 2, which gives $\sum b_A^j = 0$. If for at least one EV $\lambda \neq 1$, then according to lemma 2: $\lambda = 1 - \sum b_A^j > 0$. Finally, if $-\hat{H}_A = \partial S(0)(-A)(I - \partial\phi_A)$ is stable, then $0 < \text{Det}(I - \partial\phi_A) = 1 - \sum b_A^j$, which by ii) proves iii). ■

Proof of theorem 3 Let $x \in Cr_A$ and $Q = \sum x_i$. As $-H_A(x) = (-A(x))(I - \partial\phi_A(x))$, where A is the negative diagonal matrix with entries $\Pi_{jj}^j(x_j, \sum x_i)$, the proof of a) parallels the proof of proposition 1. To see b) and c) note that $\text{Det}(-H_A(x)) > 0$ iff $\sum_j b_A^j(Q) < 1$ by lemma 2, and both claims follow from proposition 5. If $b_A^j(Q(x)) \geq 0$ on Cr_A then, by proposition 5, contraction-stability implies CTBR stability for homogeneous diagonal S , and as $-H_A(x)$ is a Z -matrix on Cr_A , $\hat{H}(x) = \partial S(0)A(x)(I - \partial\phi_A(x))$ and $\check{H}_A(x) = \partial S(0)(\partial\phi_A(x) - I)$ we may use propositions 3 and 2, as in the proof of theorem 2, to establish that ii) \Rightarrow iii) \Leftrightarrow iv), and iv) \Rightarrow i) follows from a). As ii) \Leftrightarrow v) is clear, d) has been proven. Finally, if $b_A^j(Q(x)) \leq 0$ on Cr_A then there is a unique ATB equilibrium x_A^* , and it follows from lemma 2 ii) that $H_A(x_A^*)$ is D -stable. Therefore e) follows from the proof of proposition 2. ■

Proof of lemma 1 The case where $\rho(A) = 0$ is trivial so suppose $\rho(A) > 0$. Note that $\alpha \in \sigma(M)$ if and only if

$$MV = \begin{pmatrix} A(v_2 + \dots + v_N) \\ A(v_1 + v_3 + \dots + v_N) \\ \vdots \\ A(v_1 + \dots + v_{N-1}) \end{pmatrix} = \alpha V \quad V \neq 0 \quad (7)$$

where $V = (v_j)_{j=1}^N$ and each v_j is itself a k -vector. If $Av = \lambda v$ then setting $v_j = \frac{v}{N-1}$ shows that $\lambda(N-1) \in \sigma(M)$, as every row of (7) reads $Av = \frac{\alpha}{N-1}v$. This is true for every $\lambda \in \sigma(A)$ (with the corresponding eigenvector), hence we have found k EV of M . To find the other EV of M , first set $v_1 = v$, $v_2 = -v$ and $v_3 = \dots = v_N = 0$. Then, the first row of (7) reads $A(-v) = \alpha v$, the second row is $Av = \alpha(-v)$ and all other rows are $0 = 0$. Hence we found

k further EV of M , and they take on the values $-\sigma(A)$. Continuing by setting $v_1 = v$ and $v_3 = -v$ leaving all other coordinates zero gives the next k EV (note that the so constructed eigenvectors of M are linearly independent). Proceeding in this manner shows that all EV of M must be either $(N - 1)\lambda$ or $-\lambda$, $\lambda \in \sigma(A)$. ■

Proof of proposition 6 The first claim follows from theorem 6 (supplementary material), lemma 1 and $\frac{\partial \bar{\varphi}(x_1^*)}{\partial \bar{x}} = (N - 1) \frac{\partial \varphi^1(x^*)}{\partial x_2}$. If $k = 1$ and $\frac{\partial \varphi^1(x^*)}{x_2} \leq -1$, then $I - \partial \phi(x^*)$ must have a non-positive EV by the proof of lemma 1. Hence x^* cannot be gradient nor CTBR stable for arbitrary homogeneous diagonal S , so suppose that $\frac{\partial \varphi^1(x^*)}{x_2} > -1$ from now on. The equivalence between gradient and CTBR stability then follows either from lemma 2 ii) for $-1 < \frac{\partial \varphi^1(x^*)}{x_2} < 0$ (as $-(I - \partial \phi(x^*))$ is D -stable) or from proposition 2 for $\frac{\partial \varphi^1(x^*)}{x_2} \geq 0$ (as $(I - \partial \phi(x^*))$ is a Z -matrix). Therefore we only have to verify that any EV λ of $(I - \partial \phi(x^*))$ is positive iff $\frac{\partial \varphi^1(x^*)}{x_2} \in (-1, \frac{1}{N-1})$. Because of lemma 2 i) $\lambda = 1 + \frac{\partial \varphi^1(x^*)}{x_2}$ or $\lambda = 1 + \frac{\partial \varphi^1(x^*)}{x_2}(1 - N)$, which completes the proof. ■

Proof of proposition 7 For NB the claim follows from the arguments in the main text and figure 1. Turning to the ATB dynamics, we set $y^t \equiv x_A^{tN}$, and sequentiality gives $y^{t+N} = \left(\varphi_A^j(y^{t+j-1}) \right)_{j=1}^N$, where $\varphi_A^j(y^{t+j-1}) = \varphi_A^j \left(\sum_{i=1}^N y_i^{t+j-1} \right)$. We change this N -th order process to a first-order system using $w_m^t \equiv y^{t+m}$ for $m = 1, \dots, N - 1$. This leaves us with the N^2 -dimensional first-order system

$$w_{N-1}^{t+1} = \begin{pmatrix} \varphi_A^1(y^t) \\ \varphi_A^2(w_1^t) \\ \vdots \\ \varphi_A^N(w_{N-1}^t) \end{pmatrix} \quad \begin{pmatrix} w_{n-2}^{t+1} \\ \vdots \\ y^{t+1} \end{pmatrix} = \begin{pmatrix} w_{n-1}^t \\ \vdots \\ w_1^t \end{pmatrix}$$

According to theorem 6 (supplementary material) the mapping $(\varphi_A^1(y), \dots, w_1)$ is a local contraction at the symmetric equilibrium x_A^* iff the spectral radius of its Jacobian $J(x_A^*)$ is

less than 1. $J(x_A^*)$ takes on the form of a partitioned matrix

$$J(x_A^*) = \begin{pmatrix} L_N & L_{N-1} & \cdots & \cdots & L_1 \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & I & 0 \end{pmatrix}$$

where every partition is $N \times N$, and L_j has zero entries except for the j th row, where all entries correspond to $a \equiv \frac{\partial \varphi_A^1(x_A^*)}{\partial x_2}$. The EV equation for $J(x_A^*)$ implies that an EV λ solves

$$a \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \lambda & \cdots & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ \lambda^{N-1} & \cdots & \cdots & \lambda^{N-1} \end{pmatrix} v = \lambda^N v \quad v = (v_1, \dots, v_N)$$

Summing up over the rows implies that $a \sum_{z=0}^{N-1} \lambda^z = \lambda^N$, or $a = \frac{\lambda^N(1-\lambda)}{1-\lambda^N}$, from which it can be verified that $|\lambda| < 1$ whenever $a \in (-1, \frac{1}{N})$. ■

Proof of proposition 8 Evaluating the adjustment matrix of ϕ^{r+1} at x^* gives $\partial \phi(x^*)^{r+1}$, fact 4 of the supplementary material establishes

$\rho(\partial \phi(x^*)^{r+1}) = \rho(\partial \phi(x^*))^{r+1}$, which by theorem 6 (supplementary material) and theorem 6 implies the result. ■

7.2 Counterexamples

We provide a sequence of examples illustrating that, in general, no stability ordering exists. The fact that contraction-stability may not imply gradient or CTBR stability can be seen from the symmetric equilibrium with substitutes (section 5.1).

Example 1: Contraction-stability or CTBR stability \nRightarrow gradient stability Let $k = 1$, $N = 3$, $S = I$ and suppose that

$$H(x^*) = \begin{pmatrix} -100 & 100 & -200 \\ 250 & -100 & -250 \\ 4 & -2 & -1 \end{pmatrix} \quad \partial\phi(x^*) = \begin{pmatrix} 0 & 1 & -2 \\ 2.5 & 0 & -2.5 \\ 4 & -2 & 0 \end{pmatrix}$$

Then x^* is contraction-stable ($\rho(\partial\phi(x^*)) = \frac{1}{\sqrt{2}}$) and also CTBR stable for homogeneous diagonal cofactor S (proposition 3), but not gradient stable (for arbitrary diagonal S), as e.g. if $S = I$ $H(x^*)$ has two positive EV's ($\lambda_1 = 55.48$, $\lambda_2 = 1.05$). ■

Example 2: Contraction-stability \nRightarrow CTBR stability (heterogeneous adj. rates)

Let $k = 1$, $N = 4$ and suppose that

$$\check{H}(x^*) = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix}}_{=S} \left(\underbrace{\begin{pmatrix} 0 & 5 & -1 & -3/2 \\ -1/2 & 0 & 1 & -2 \\ -2/5 & -7/4 & 0 & 0 \\ -3/4 & -3/2 & -1 & 0 \end{pmatrix}}_{=\partial\phi(x^*)} - I \right)$$

Then x^* is contraction-stable (as $\rho(\partial\phi(x^*)) = 0.845$) but not CTBR stable (as $\check{H}(x^*)$ has an EV equal to $6.133 > 0$). ■

Example 3: Gradient stability \nRightarrow CTBR stability Let $k = 1$, $N = 3$, $S = I$ and suppose that

$$H(x^*) = \begin{pmatrix} -5 & 0 & -5 \\ -3 & -1 & -3 \\ 20 & -30 & -10 \end{pmatrix} \quad \check{H}(x^*) = \begin{pmatrix} -1 & 0 & -1 \\ -3 & -1 & -3 \\ 2 & -3 & -1 \end{pmatrix}$$

Then x^* is gradient stable (as $\hat{H}(x^*) = H(x^*)$ has eigenvalues with real parts = $\{-3, -3, -10\}$) but not CTBR stable (as $\check{H}(x^*)$ has two complex EV's with real part = 0.57). ■

Example 4: H stable $\not\Rightarrow H$ D -stable in sum-aggregative games Let $N = 4$ and consider (note that $b_j(x) > -1$ as is naturally the case in the Cournot model)

$$H = \begin{pmatrix} -5 & 3 & 3 & 3 \\ 4 & -5 & 4 & 4 \\ 4 & 4 & -5 & 4 \\ -4 & -4 & -4 & -12 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.01 \end{pmatrix}$$

Then $\sigma(H) = \{-9, -8, -7.83, -2.17\}$ but $\sigma(SH) = \{-9, -8.29, 2.09, 0.08\}$. ■

Example 5 This example shows that in a one-dimensional non sum-aggregative game with complements the uniqueness-condition need not imply the contraction-condition. Let

$$\partial\phi(x) = \begin{pmatrix} 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix} \geq 0$$

Then $\sigma(\partial\phi(x)) = \{2.425, -2.135, -1.472, 1.181\}$, which shows that as $\text{Det}(-H(x)) = 2 > 0$ the uniqueness-condition is satisfied at x , but not the contraction-condition.

Example 6 Finally, we present an example illustrating that theorem 4 does not generally extend beyond $k = 1$. If $k = N = 2$ and

$$\frac{\partial\tilde{\varphi}(x_1^*)}{\partial\bar{x}} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

then $\rho(\sigma(\frac{\partial\tilde{\varphi}(x_1^*)}{\partial\bar{x}})) = 3$ but $\text{sign}(\text{Det}(-\check{H}(x_1^*))) = \text{sign}((1-3)(1-3)) > 0$. Hence while the uniqueness condition is satisfied, the contraction-condition fails to hold despite a positive slope matrix. Similar counterexamples for the other dynamics can be constructed.

Acknowledgements I wish to thank Diethard Klatte and Armin Schmutzler as well as participants at seminars in Zurich and at the UECE Lisbon Theory meeting 2011 for their valuable comments.

References

- Al Nowaihi, A. and Levine, P. L. (1985). The stability of the cournot oligopoly model: a reassessment. *Journal of Economic Theory*, 35(2):307–321.
- Alos-Ferrer, C. and Ania, A. B. (2005). The evolutionary stability of perfectly competitive behavior. *Economic Theory*, 26:497–516.
- Avriel, M., Diewert, W. E., and Zang, I. (1981). Nine kinds of quasiconcavity and concavity. *Journal of Economic Theory*, 25:397 – 420.
- Cournot, A. (1897). *Researches into the Mathematical Principles of the Theory of Wealth*. New York: McMillan.
- Dastidar, K. G. (2000). Is a unique cournot equilibrium locally stable? *Games and Economic Behavior*, 32:206–218.
- Dindos, M. and Mezzetti, C. (2006). Better-reply dynamics and global convergence to nash equilibrium in aggregative games. *Games and Economic Behavior*, 54(2):261–292.
- Dixit, A. (1986). Comparative statics for oligopoly. *International Economic Review*, 27:107–122.
- Furth, D. (1986). Stability and instability in oligopoly. *Journal of Economic Theory*, 40(2):197–228.
- Grossmann, M., Hefti, A. M., and Lang, M. (2012). Aggregative contests and ex-post heterogeneity. *ISU working paper 161*.
- Hadar, J. (1966). Stability of oligopoly with product differentiation. *The Review of Economic Studies*, 33:57–60.
- Hefti, A. M. (2013). Uniqueness and stability in symmetric games. *ECON working paper*.
- Hershkowitz, D. (1992). Recent directions in matrix stability. *Linear Algebra and its Applications*, 171:161–186.
- Ho, T.-H., Camerer, C., and Weigelt, K. (1998). Iterated dominance and iterated best response in experimental p-beauty contests. *The American Economic Review*, 88(4):947–969.
- Hommes, C. (2013). Reflexivity, expectations feedback and almost self-fulfilling equilibria: economic theory, empirical evidence and laboratory experiments. *Journal of Economic Methodology*, 20(4):406–419.
- Horn, R. A. and Johnson, C. R. (1985). *Matrix Analysis*. Cambridge University Press.
- Hosomatsu, Y. (1969). A note on the stability conditions in cournot’s dynamic market solution when neither the actual market demand function nor the production levels of rivals are known. *Review of Economic Studies*, 36:117–122.
- Huck, S., Normann, H.-T., and Oechssler, J. (2002). Stability of the cournot process: experimental evidence. *International Journal of Game Theory*, 31:123–136.
- Konrad, K. (2009). *Strategy and Dynamics in Contests*. Oxford University Press.
- Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995). *Microeconomic theory*, volume 1. Oxford university press New York.
- Moulin, H. (1984). Dominance solvability and cournot stability. *Mathematical Social Science*, 7:83–102.
- Moulin, H. (1986). *Game theory for the social sciences*. NYU press.

- Okuguchi, K. and Yamazaki, T. (2008). Global stability of unique nash equilibrium in cournot oligopoly and rent-seeking game. *Journal of Economic Dynamics and Control*, 32:1204–1211.
- Poincare, H. (1905). *Science and hypothesis*. London: Walter Scott Publishing.
- Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica*, 33:520–534.
- Scarf, H. (1960). Some examples of global instability of the competitive equilibrium. *International Economic Review*, 1(3):pp. 157–172.
- Vives, X. (1999). *Oligopoly pricing*. Cambridge MIT Press.

8 Supplementary material (for online publication)

The purpose of this mathematical appendix is to provide a compact and short exposition of the contraction principle.³¹ We further discuss important sufficient conditions for contractions (in one- and higher-dimensional cases).

Essential facts from matrix analysis Let M_n denote the set of all real $n \times n$ matrices. Matrix norms are denoted by $\|\cdot\|$. A matrix norm has all properties of a vector norm and additionally satisfies submultiplicativity (i.e. $\|AB\| \leq \|A\| \|B\|$). Submultiplicativity matters as then the spectral radius is a lower bound for any matrix norm.³² An important class of matrix norms are those induced by a vector norm $|\cdot|$: $\|A\|_{|\cdot|} \equiv \max_{|v|=1} |Av|$. As $\|A\|_{|\cdot|}$ turns the space of all bounded linear operators from X to Y into a Banach space (as Y is a Banach space), such norms are frequently referred to as operator norms. The following facts about matrix norms and their spectra are known (see Horn and Johnson (1985)):

Fact 3 For any matrix norm $\|\cdot\|_L$ there exists an operator norm $\|\cdot\|_{|\cdot|}$ such that $\|A\|_{|\cdot|} \leq \|A\|_L$ for any $A \in M_n$. Moreover, $|Av| \leq \|A\|_{|\cdot|} |v|$ holds for any $A \in M_n$ and $v \in X$.

Fact 4 If $A \in M_n$ then $\rho(A) \leq \|A\|$ for any matrix norm. Moreover, for any $A \in M_n$ and any $\varepsilon > 0$ there exists a matrix norm such that $\|A\| < \rho(A) + \varepsilon$. Finally, if $t \in \mathbb{N}$ then $\rho(A^t) = \rho(A)^t$.

Lipschitz functions $X = (\mathbb{R}^n, |\cdot|_X)$ and $Y = (\mathbb{R}^m, |\cdot|_Y)$ are two complete metric spaces. We will be concerned with compact and convex subsets³³ \bar{U} of \mathbb{R}^n with non-empty interior U . Note that $(\bar{U}, |\cdot|_{X|\bar{U}})$ is another complete metric space. We identify a metric space by its underlying set, i.e. we set $\bar{U} \equiv (\bar{U}, |\cdot|)$. Among continuous functions, the subclass of Lipschitz-continuous functions plays a major role for economic theory. A function $\phi: \bar{U} \rightarrow \mathbb{R}^m$ is Lipschitz on \bar{U} if there is $q > 0$ such that $|\phi(x) - \phi(x')|_Y \leq q|x - x'|_X$ for any $x, x' \in \bar{U}$. Lipschitz-continuity arises naturally in many economic and game-theoretic applications, as continuously differentiable functions on convex sets are locally Lipschitz, and Lipschitz if these sets are also compact.

³¹See e.g. Moulin (1984) for a classical proof in the one-dimensional case.

³²Mathematically, this holds as submultiplicativity imposes a restriction on which linear combinations of matrix norms generate new matrix norms. E.g. if $\|\cdot\|$ is a matrix norm, then $r\|\cdot\|$ is a matrix norm if and only if $r \in [1, \infty)$.

³³Any reference to open or closed subsets of \bar{U} means open or closed relative to \bar{U} in the usual topological sense.

Fact 5 *If in a game $\phi(\mathcal{W}) \subset \text{Int}(\mathcal{W})$ and \mathcal{W} is compact, then ϕ is Lipschitz.*

Proof: Because $\phi(\mathcal{W}) \subset \text{Int}(\mathcal{W})$, the IFT asserts that ϕ is locally represented by a continuously differentiable function, which in turn implies ϕ to be locally Lipschitz. Consequently, ϕ is Lipschitz if \mathcal{W} is compact. ■

Contractions ϕ is called a contraction (mapping) if it is Lipschitz with $q < 1$. The set of all contractions from X to Y are denoted by $\mathcal{C}(X, Y)$. It is important to bear in mind that contractions are defined contingent on certain norms (as X, Y are spaces rather than just sets), and the contraction property generally is not invariant under equivalent norms³⁴. Contractions are the main ingredient of the Banach FP theorem. Its beauty stems from the fact that it asserts three desirable properties of a game - existence and uniqueness of equilibrium as well as global stability of the best-reply map - to occur simultaneously: If $\phi \in \mathcal{C}(\mathcal{W}, \mathcal{W})$ and \mathcal{W} is a complete metric space (e.g. \mathcal{W} is a closed subset of X), then ϕ has exactly one FP x^* and the recurrence relation $x^t = \phi(x^{t-1})$ converges³⁵ to x^* for any initial value $x^0 \in \mathcal{W}$.

8.1 Characterization of (local) contractions

The following theorem provides a characterization for a differentiable mapping ϕ to be a contraction.

Theorem 5 *Suppose $\phi \in C^0(\bar{U}, \mathbb{R}^m)$ is (Frechet)-differentiable on U . Then there exists a norm $|\cdot|$ on \bar{U} such that $\phi \in \mathcal{C}(\bar{U}, Y)$ if and only if*

$$\sup_{x \in U} \|\partial\phi(x)\|_{|\cdot|} < 1 \quad (8)$$

Proof: "⇒". Let $q = \sup_{x \in U} \|\partial\phi(x)\|_{|\cdot|} < 1$. Because U is an open, convex set, the mean value theorem implies the following bound for any $x, x' \in U$:

$$\begin{aligned} |\phi(x) - \phi(x')| &\leq \sup_{0 \leq t \leq 1, |v|=1} (|\partial\phi(x + t(x' - x)) \cdot v|) |x - x'| \\ &\leq q|x - x'| \end{aligned} \quad (9)$$

³⁴This is a difference to the more general Lipschitz property, which is preserved under equivalent norms.

³⁵In memoriam of its initial discoverer, such convergence of the joint best-reply has frequently been quoted as Cournot stability.

Hence ϕ is a contraction on U . Let $x \in \partial\bar{U}$, $x' \in \bar{U}$ and take any two sequences $(x_n), (x'_n)$ in U with $x_n \rightarrow x$ and $x'_n \rightarrow x'$. Because of $|\phi(x_n) - \phi(x'_n)| \leq q|x_n - x'_n|$ continuity implies $|\phi(x) - \phi(x')| \leq q|x - x'|$ which shows that ϕ is a contraction on \bar{U} .

" \Leftarrow ". Suppose $\exists q < 1$ such that $|\phi(x) - \phi(x')| \leq q|x - x'| \forall x, x' \in \bar{U}$. Take an arbitrary $x \in U$ and an arbitrary $v \in \mathbb{R}^n$ with $|v| = 1$. Then there exists $\varepsilon > 0$ such that $x + tv \in U$ for $t \in (-\varepsilon, \varepsilon)$, and $q \geq \frac{|\phi(x+tv) - \phi(x)|}{|t|}$ for $t \neq 0$. As ϕ is Frechet-differentiable on U , the directional derivatives exist. Hence

$$q \geq \lim_{t \rightarrow 0} \frac{|\phi(x+tv) - \phi(x)|}{|t|} = \left| \lim_{t \rightarrow 0} \left(\frac{\phi(x+tv) - \phi(x)}{t} \right) \right| = |\partial\phi(x) \cdot v|$$

As both $x \in U$ and v where arbitrary (up to $|v| = 1$), we get $\sup_{x \in U, |v|=1} |\partial\phi(x) \cdot v| = \sup_{x \in U} \|\partial\phi(x)\|_{|\cdot|} \leq q < 1$. ■

Theorem 5 says that ϕ is a contraction from the space \bar{U} to Y if and only if its directional derivatives, i.e. its local rates of change in some direction v , are bounded by one as measured by the $|\cdot|_Y$ -norm for any point $x \in U$. Note that ϕ is not required to be continuously differentiable, nor differentiable at boundary points. In terms of minimal assumptions the proof of theorem 5 shows that compactness of \bar{U} is not critical, but convexity is a vital assumption. By choosing adequate norms, Lipschitz functions can be made contractive. In particular, if $\phi : \bar{U} \rightarrow \mathbb{R}^m$ is Lipschitz with Lipschitz-constant q and if $\tilde{Y} = (\mathbb{R}^m, |\cdot|_{\tilde{Y}})$ with $|\cdot|_{\tilde{Y}} = \frac{1}{1+q} |\cdot|_Y$, then $\phi \in \mathcal{C}(\bar{U}, \tilde{Y})$. It is the tightening of this freedom in choosing an appropriate Y -norm (or X -norm) from which the Banach FP theorem gets much of its bite: By presumption, we are only at liberty to choose one norm for both the domain and the codomain of ϕ . If $\phi : \bar{U} \rightarrow \bar{U}$ and the premise of theorem 5 is satisfied, then fact 3 and theorem 5 assert that there is a norm $|\cdot|$ such that $\phi \in \mathcal{C}(\bar{U}, \bar{U})$ if and only if there is a matrix norm $\|\cdot\|$ with $\|\partial\phi(x)\| < 1$ for any $x \in U$. Under the additionally assumption that ϕ is even continuously differentiable on \bar{U} , we obtain a condition on the spectral radius of $\partial\phi(x)$:

Corollary 3 *Suppose that $\phi \in C^1(\bar{U}, \bar{U})$. Then there exists a norm $|\cdot|$ on \bar{U} such that $\phi \in \mathcal{C}(\bar{U}, \bar{U})$ if and only if $\sup_{x \in \bar{U}} \rho(\partial\phi(x)) < 1$.*

Proof: " \Rightarrow " Let $\sup_{x \in \bar{U}} \rho(\partial\phi(x)) = \delta < 1$ and note that continuity of $\rho(\partial\phi(x))$ implies that $\exists \varepsilon > 0$ such that $\rho(\partial\phi(x)) < \delta + \varepsilon < 1$ holds on \bar{U} . Then, by fact 4, $\forall x_0 \in \bar{U}$ there exists a matrix norm $\|\cdot\|_{(x_0)}$ such that $\|\partial\phi(x_0)\|_{(x_0)} < \delta + \varepsilon$. Continuity of $\partial\phi(x)$ asserts the existence

of an open neighborhood $B(x_0) \subset \bar{U}$ such that $\|\partial\phi(x)\|_{(x_0)} < \delta + \varepsilon$ for any $x \in B(x_0)$. Because \bar{U} is compact and $\bigcup_{x \in \bar{U}} B(x)$ covers \bar{U} there exists a finite subcover $\bigcup_{j=1}^n B(x_j)$, and $\|\partial\phi(x)\|_{(x_j)} < \delta + \varepsilon$ for any $x \in B(x_j)$. Further, $\|\cdot\| \equiv \max\{\|\cdot\|_{(x_1)}, \dots, \|\cdot\|_{(x_n)}\}$ is a matrix norm such that $\|\partial\phi(x)\| < \delta + \varepsilon$ holds for any $x \in \bar{U}$ and the claim follows from fact 3 and theorem 5.

" \Leftarrow " Follows from theorem 5 and fact 4. ■

We now turn to the local version of theorem 5. Suppose that $\phi : \bar{U} \rightarrow \bar{U}$ and let x^* be a FP of ϕ . The FP x^* is contraction-stable if ϕ induces a local contraction about x^* , i.e. if there is a convex, complete metric space $(V, |\cdot|)$, $x^* \in V \subset \bar{U}$, such that $\phi|_V \in \mathcal{C}(V, V)$.

Theorem 6 (Local contractions) *Suppose $\phi : \mathcal{W} \rightarrow \mathcal{W}$, $\phi(x^*) = x^*$, $\partial\phi(x^*)$ exists and $\partial\phi$ is continuous at x^* . Then there is a neighborhood V about x^* such that $\phi(V) \subset V$ and ϕ is a contraction on V if and only if there is a matrix norm $\|\cdot\|$ such that*

$$\|\partial\phi(x^*)\| < 1 \quad \text{or equivalently} \quad \rho(\partial\phi(x^*)) < 1 \quad (10)$$

Then, the best-reply process $x^t = \phi(x^{t-1})$ converges locally to x^ , and $\lim_{t \rightarrow \infty} h(x^t) = h(x^*)$ for any function $h(x)$ that is continuous at x^* .*

Proof: " \Rightarrow " Let $\sigma(x) = \|\partial\phi(x)\|$. Hence $q \equiv \sigma(x^*) < 1$. As σ is continuous at x^* there exists $\delta > 0$ and a closed ball $\bar{\mathbb{B}}(x^*, \delta) \equiv V$ such that $\sigma(x) < 1 \forall x \in V$. Let $Q \equiv \sup_{x \in V} \sigma(x) < 1$. Then fact 3 and theorem 5 assert that ϕ is a contraction on V , i.e. $|\phi(x) - \phi(x')| \leq Q|x - x'|$ for $x, x' \in V$, and $|\phi(x) - \phi(x^*)| \leq Q\delta < \delta$ for $x \in V$ shows that $\phi(V) \subset V$. " \Leftarrow " Follows from theorem 5, and the last two claim follow from the Banach FP theorem and continuity. Finally, the norm condition can be replaced by the spectral radius condition as a consequence of fact 4. ■

Note that if $x^t \rightarrow x^*$ with $x^0 \neq x^*$ and $\rho(\partial\phi(x^*)) \geq 1$, ϕ cannot be linear around x^* :

Corollary 4 *x^* is contraction-stable if and only if the linearization of ϕ at x^* , $L(x) = \partial\phi(x^*)x + (I - \partial\phi(x^*))x^*$, is a contraction on \mathbb{R}^n .*

Proof: " \Rightarrow " By presupposition there is $|\cdot|$ and $q < 1$ such that $|L(x) - L(x')| \leq q|x - x'|$ for any $x, x' \in \mathbb{R}^n$ or equivalently $|\partial\phi(x^*)v| \leq q|v|$, $v \in \mathbb{R}^n$. Hence also $\|\partial\phi(x^*)\|_{|\cdot|} \leq q < 1$, and

the claim follows from theorem 6. "⇐" If x^* is contraction-stable, then $\|\partial\phi(x^*)\|_{|\cdot|} < 1$ for some $|\cdot|$ because of theorem 6. The claim then follows from $|L(x) - L(x')| = |\partial\phi(x^*)(x - x')| \leq \|\partial\phi(x^*)\| |x - x'|$. ■

The spectral radius not only qualitatively determines if a FP is contraction-stable, but also influences how quickly a FP is approached; a smaller value means that fewer iterations are required. The linearization produces an estimate on the number of adjustments it takes to approach a certain vicinity of the equilibrium. Suppose that $\rho(\partial\phi(x^*)) \neq 0$, $x^0 \neq x^*$ and set $x^t = L(x^{t-1}) \approx \phi(x^{t-1})$. This gives $x^t - x^* = \partial\phi(x^*)^t(x^0 - x^*)$ by backwards induction. Then fact 4 implies that for any arbitrarily small $\varepsilon > 0$ there is a matrix norm $|\cdot|$ such that $d \equiv \frac{|x^t - x^*|}{|x^0 - x^*|} < \rho(\phi(x^*))^t + \varepsilon$, where d is the fraction of distance left to the equilibrium after t iterations. Hence $t = \left\lfloor \frac{\ln(d)}{\ln(\rho(\partial\phi(x^*)))} + 1 \right\rfloor$ is the approximate number of adjustments required to cover at least $1 - d$ of the initial distance to the equilibrium.

8.2 Sufficient conditions for contraction-stability

A straightforward application of the IFT shows that, for $k = 1$, the Hadar-condition $\|\partial\phi(x)\|_{|\cdot|_\infty} < 1$ is equivalent to the requirement that $H(x)$ has a dominant negative diagonal. Corollary 2 in section 5.1 shows that diagonal dominance is in fact necessary and sufficient for symmetric equilibria of symmetric games to be contraction-stable (and locally dominance-solvable).

The equivalence between diagonal dominance and the Hadar-condition breaks down if $k > 1$. It is straightforward to construct examples already if $k = N = 2$, where $R_m(x^*) < 1$ is true for all m but $H(x^*)$ violates diagonal dominance. Nevertheless, diagonal dominance of $H(x^*)$ implies the Hadar-condition (and thus also local contraction-stability):

Corollary 5 *Suppose $k = 2$, x^* is a FP and $H(x^*)$ has a dominant negative diagonal. Then $R_m(x^*) < 1$, $m = 1, \dots, 2N$.*

Proof: For a vector v let $v^+ \equiv (|v_i|)$ denote the vector of the absolute values of the components of v . Similarly, if M is a matrix then M^+ denotes the matrix of absolute values of the components of M . The triangle inequality implies $(Mv)^+ \leq M^+v^+$. Let $A_g = \frac{\partial^2 \Pi^g(x)}{\partial x_g \partial x_g}$ denote the Hessian of $\Pi^g(x)$. The IFT gives for $j \neq g$ and $1 \leq i \leq 2$:

$$\left(\frac{\partial \varphi^g}{\partial x_{ji}} \right)^+ = \left(-(A_g)^{-1} \left(\frac{\partial \nabla_g \Pi^g}{\partial x_{ji}} \right) \right)^+ \leq \left((A_g)^{-1} \right)^+ \left(\frac{\partial \nabla_g \Pi^g}{\partial x_{ji}} \right)^+$$

which further implies

$$\sum_{j \neq g} \sum_{i=1}^k \left(\frac{\partial \varphi^g}{\partial x_{ji}} \right)^+ \leq \left((A_g)^{-1} \right)^+ z_g \quad (11)$$

where $z_g = \sum_{j \neq g} \sum_{i=1}^2 \left(\frac{\partial \nabla_g \Pi^g}{\partial x_{ji}} \right)^+$. Let \hat{A}_g be the matrix derived from A_g by taking the absolute values of each off-diagonal element. Because $H(x^*)$ has a dominant negative diagonal it follows that $-\hat{A}_g \cdot 1 > z_g$, and

$$\left((A_g)^{-1} \right)^+ = \left(-\hat{A}_g \right)^{-1} \geq 0 \quad (12)$$

Hence we get

$$\left(-\hat{A}_g \right)^{-1} \left(-\hat{A}_g \right) \cdot 1 > \left(-\hat{A}_g \right)^{-1} z_g \geq \left((A_g)^{-1} \right)^+ z_g$$

which by (11) gives $I \cdot 1 > \sum_{j \neq g} \sum_{i=1}^2 \left(\frac{\partial \varphi^g}{\partial x_{ji}} \right)^+$ for any $g = 1, \dots, N$ showing that $R_m < 1$ holds $\forall m$. ■

Remark: It is easy to generalize the above proof to the case $k > 2$ up to expression (12).

Using laborious Laplace expansions (and the triangle inequality) it can be shown that

$$\left((A_g)^{-1} \right)^+ \leq \left(\left(-\hat{A}_g \right)^{-1} \right)^+ = \left(-\hat{A}_g \right)^{-1}$$

The equality between the second and third term follows from the fact that $-\hat{A}_g$ is a diagonally dominant matrix with non-positive off-diagonal elements (a M -matrix) and it is known that then $\left(-\hat{A}_g \right)^{-1}$ is positive.

8.3 Convergence of the average

We provide a simple example, where convergence of the average implies convergence of best-replies almost surely. Consider a symmetric one-dimensional linear game with $\varphi^j(x_{-j}) = a \sum_{i \neq j} x_i + b$ and symmetric equilibrium $x^* = (x_1^*, \dots, x_1^*)$ satisfying $x_1^* = a(N-1)x_1^* + b$. Let

$h^t = \frac{\sum x_j^t}{N}$ and $z^t = h^t - h^*$, where obviously $h^* = x_1^*$. Then

$$\begin{aligned} z^t &= \frac{1}{N} \sum \varphi^j(x_{-j}^{t-1}) - x_1^* = a(N-1)h^{t-1} + b - x_1^* \\ &= a(N-1)(z^{t-1} + x_1^*) + b - x_1^* = a(N-1)z^{t-1} \\ &= [a(N-1)]^t \left(\frac{1}{N} \sum x_j^0 - x_1^* \right) \end{aligned}$$

Now observe that $h^t \rightarrow h^*$ but simultaneously $x^t \not\rightarrow x^*$ is possible only if $\frac{1}{N} \sum x_j^0 = x_1^*$, which does not occur almost surely as $\{x \in \mathbb{R}^n : \frac{1}{N} \sum x_j = x_1^*\}$ is a zero-measure Lebesgue set.