University of Zurich

University of Zurich

Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN 1664-705X (online)

Working Paper No. 461

Hotelling Meets Shaked and Sutton: A Unified Linear Model of Product Differentiation

Armin Schmutzler

December 2024

Hotelling Meets Shaked and Sutton: A Unified Linear Model of Product Differentiation

Armin Schmutzler*

November 2024

Abstract

This paper provides a simple unified discrete-choice framework for analyzing differentiated duopolies. This framework nests models of horizontal and vertical differentiation, including standard textbook models (Hotelling and Shaked-Sutton). Contrary to these models, it also applies to economic environments where horizontal differentiation coincides with positive correlation of product valuations across consumers, and environments where vertical differentiation coincides with negative correlation. The paper provides an equilibrium characterization that is applicable independently of the type of differentiation and the sign of the valuation correlation.

JEL Codes: D43, L13
Keywords: Duopoly, Differentiated Products, Price Competition

*Department of Economics, University of Zurich and CEPR. Email: armin.schmutzler@econ.uzh.ch. I am grateful to Andreas Hefti, Igor Letina, Nick Netzer, Yossi Spiegel, Jakub Steiner and Flavio Toxvaerd for helpful discussions.

1 Introduction

Duopoly theory distinguishes between models of horizontal and vertical differentiation. Under horizontal differentiation, each of the two products faces positive demand at equal prices, whereas under vertical differentiation all consumers prefer the same product to the alternative. The simplest textbook models are the Hotelling (1929) model of horizontal differentiation and the Shaked and Sutton (1982) model of vertical differentiation. However, these models also differ from each other in another way: The Hotelling model corresponds to situations with strictly negative correlation between the values of the two products, whereas the correlation is strictly positive for the model of Shaked and Sutton.[1] This linkage between the type of product differentiation and the correlation of valuations appears to have received little attention. This most likely reflects the fact that, presumably for historical reasons, the models are usually presented in very different ways, impeding direct comparison.[2]

This paper presents a simple discrete-choice framework for the analysis of differentiated duopoly that is not only rich enough to nest the two standard textbook models but, in addition, separates the type of differentiation from the type of correlation. In this framework, the *Unified Linear Model* (ULM) of product differentiation, the two firms both face a continuum of consumers. Each consumer is characterized by a two-dimensional valuation vector, with each component corresponding to the willingness-to-pay (WTP) for one of the two products. To maintain the simplicity of the standard textbook models, I focus on the case that (i) the support of the valuation distribution is a straight line in the two-dimensional valuation space, (ii) the distribution is uniform and (iii) marginal costs are constant. Though these assumptions are restrictive, they are compatible with several interesting cases. The standard Hotelling model with linear transportation costs corresponds to the case where the support is downward-sloping and intersects the 45-degree line. The support of the Shaked-Sutton model is an upward-sloping line which lies entirely on one side of the 45-degree line (where both valuations are equal). Importantly, however, the framework also applies to circumstances that are not commonly analyzed, even though they do not appear to be less relevant than the two standard textbook cases: Horizontal differentiation with positive correlation can be represented by a valuation distribution with an upward-sloping support that crosses the 45-degree line, while vertical differentiation with negative correlation requires a downward-sloping support that remains on one side of the 45-degree line.

The paper characterizes the unique pure-strategy equilibrium of the ULM for all four cases simultaneously, allowing for substantial cost and demand asymmetries. As is standard in the analysis of the underlying textbook models, parameter restrictions are necessary to guarantee existence of an equilibrium in which (i) both firms sell positive quantities and (ii) each consumer buys one of the two products. With these assumptions in place, one can calculate the equilibrium of the game as a function of four key parameters, namely the marginal costs of the two firms and the difference in the valuations between the two products for the two most extreme consumers.

[1]The statement about the Hotelling model refers to the standard case where firms are located at the end of the interval where consumers are located (see Section 3.1)

[2]In typical textbook treatments such as Tirole (1989) and Belleflamme and Peitz (2010), the horizontal differentiation models of Hotelling (1929) and Salop (1979) are presented as locational models with consumers who face transportation costs, while the models of vertical differentiation are location-free.

The framework not only allows us to calculate the equilibrium for standard textbook models as well as some less standard but equally natural models in a unified way, it is also helpful to understand the comparative statics, which is relevant for applications to topics such as innovation. Some comparative statics results hold no matter whether differentiation is horizontal or vertical and whether correlation is positive or negative, while for other results these distinctions are important.

The effect of a marginal cost reduction of one firm on equilibrium prices belongs to the former category: As a firm's cost falls, it lowers its prices. Reflecting strategic complementarities, the competitor follows suit, but to a lesser extent. As a result, its equilibrium market share and profits both increase, whereas they decrease for the competitor. The effects of a *uniform* change in the difference between the WTPs of the two firms is similarly robust: If the difference between the WTP for product A and product B increases by the same amount for all consumers, then the equilibrium price of firm A increases and the price of B falls. Moreover, the market share and profits of A increase; conversely for B. These effects are entirely independent of the type of differentiation and correlation.

The effects are more subtle for WTP increases that are heterogeneous across consumers. Consider a firm (say firm A) which has some consumers that prefer its product to the competitor's at equal prices. First, suppose that the difference between the WTP for product A and product B increases for the keenest consumers of firm A, thus intensifying consumer heterogeneity. Then equilibrium prices and margins of both firms increase: The change softens competition as the differences between the two firms become more pronounced. Assume for simplicity that the firms have symmetric marginal costs. Then, under horizontal differentiation, this price effect is accompanied by an increase in firm A's market share. Perhaps surprisingly, however, under vertical differentiation, the market share of firm A does not increase: Instead, this firm prefers to exploit its consumers by increasing prices so much that its demand falls. Its profits increase as the latter effect dominates the former.

Now suppose instead that firm A marginally improves its appeal to those consumers who like its products least. In other words, under vertical differentiation, the premium that these consumers are prepared to pay for product A increases further; under horizontal differentiation, the premium that these consumers are prepared to pay for product B becomes smaller. Then equilibrium prices and margins of both firms decrease as consumers become more similar and competition intensifies. Contrary to the previous case, the market share of firm A increases no matter whether competition is horizontal or vertical. Despite the reduction in margins, the increasing market share may increase its profits, but only if it is sufficiently strong relative to the competitor.

In summary, the simple model provides a unified framework to analyze product differentiation, no matter whether differentiation is horizontal or vertical and whether valuations are positively or negatively correlated. The comparative statics of prices and margins are the same for both types of differentiation. The main differences arise for the effects of biased WTP increases on market shares and profits. For instance, with symmetric marginal costs, a WTP increase that is biased towards the consumers with strongest preferences for firm A increases its market share only for horizontal differentiation—for vertical differentiation the competitor's market share increases. In contrast, a WTP increase that is biased towards consumers with strongest preferences for firm B increases the profits of firm A only under vertical differentiation, not under horizontal differentiation (still assuming symmetric costs).

The framework has the added benefit that it is straightforward to apply to two-stage models (for instance, models of advertising and innovation). The companion paper (Schmutzler, 2024) relies on the current model to analyze process, product and environmental innovation in a duopoly with a brown (polluting) and a green (less polluting) firm. Moreover, the framework is suitable for treating the mode of competition as endogenous: For instance, firms may engage in process or product innovations to turn horizontal differentiation into vertical differentiation. Similarly, under vertical differentiation, a laggard may want to invest to turn the mode of competition into horizontal differentiation. Standard models of product differentiation do not allow for such considerations as they typically treat the mode of competition as exogenously fixed, either horizontal or vertical differentiation.

The model is related to Perloff and Salop (1985) who also consider the distribution of consumer valuations as a primitive of the model. Importantly, however, their model assumes independent distributions for different products, while my paper emphasizes the existence of correlations between the valuations for different products.

In Section 2, I introduce the model. Section 3 shows how the assumptions of the model can be derived from familiar textbook models or alternatively from a model where products differ in an objective and a subjective component. Section 4 contains the main results. Section 5 concludes. Proofs and technical details are in the appendix.

2 The Model

Firms $i \in \{A, B\}$ produce differentiated products, also denoted as i. Marginal costs are constant, given as $c_i \geq 0$. The firms simultaneously set prices p_i. Consumers decide which of the two products (if any) to buy. There is a unit mass of consumers who are distributed on the interval $[0, 1]$, with corresponding atomless probability measure μ. Unless stated otherwise, I assume that this distribution is uniform. Consumer $k \in [0, 1]$ values product i at $v_k^i \in \mathbb{R}_+$. If consumer k buys at all at prices (p_i, p_j), he buys from firm i if $v_k^i - p_i > v_k^j - p_j$ for $j \neq i$. The following notation will be used throughout the paper:

$$\Delta_C := c_A - c_B$$
$$\Delta_p := p_A - p_B$$
$$\Delta_k := v_k^A - v_k^B$$

Using the new notation and assuming full coverage, demand functions become[3]

$$x_A(p_A, p_B) = \mu\{k \in [0, 1] | \Delta_k > \Delta_p\};$$
$$x_B(p_B, p_A) = \mu\{k \in [0, 1] | \Delta_k < \Delta_p\}.$$

The profits of firm $i \in \{A, B\}$ are

$$\pi_i(p_i, p_j) = (p_i - c_i)x_i(p_i, p_j).$$

The first assumption states that the support of the valuation distribution in $\mathbb{R}_+^2$ is a straight line from (v_0^A, v_0^B) to (v_1^A, v_1^B).

[3]The assumptions below will make sure that full coverage arises in equilibrium and after any unilateral deviation.

Assumption 1. *For each $k \in [0,1]$ and $i \in \{A, B\}$, valuations satisfy*

$$v_k^i = v_0^i + k\left(v_1^i - v_0^i\right)$$

Using Assumption 1 and uniformity, the distribution of consumer valuations is determined by the valuations of the extreme consumer types, (v_0^A, v_0^B) and (v_1^A, v_1^B). Moreover,

$$\Delta_k = \Delta_0 + k\left(\Delta_1 - \Delta_0\right). \tag{1}$$

The following assumption imposes restrictions on valuations.

Assumption 2. *(i) $v_1^A > v_0^A$;*

(ii) $\Delta_1 > \Delta_0$ or, equivalently, $v_1^A - v_1^B > v_0^A - v_0^B$;

(iii) $\Delta_1 = v_1^A - v_1^B > 0$.

Part (i) states that $k = 1$ has higher valuation for product A than $k = 0$ so that Assumption 1 implies more generally that v_k^A is increasing in k. Together, part (ii) and Assumption 1 imply the single-crossing condition that $v_k^A - v_k^B$, the difference in WTPs, is increasing in k. (iii) requires that at least the consumer with the strongest valuation for product A prefers it to the alternative B at equal prices.

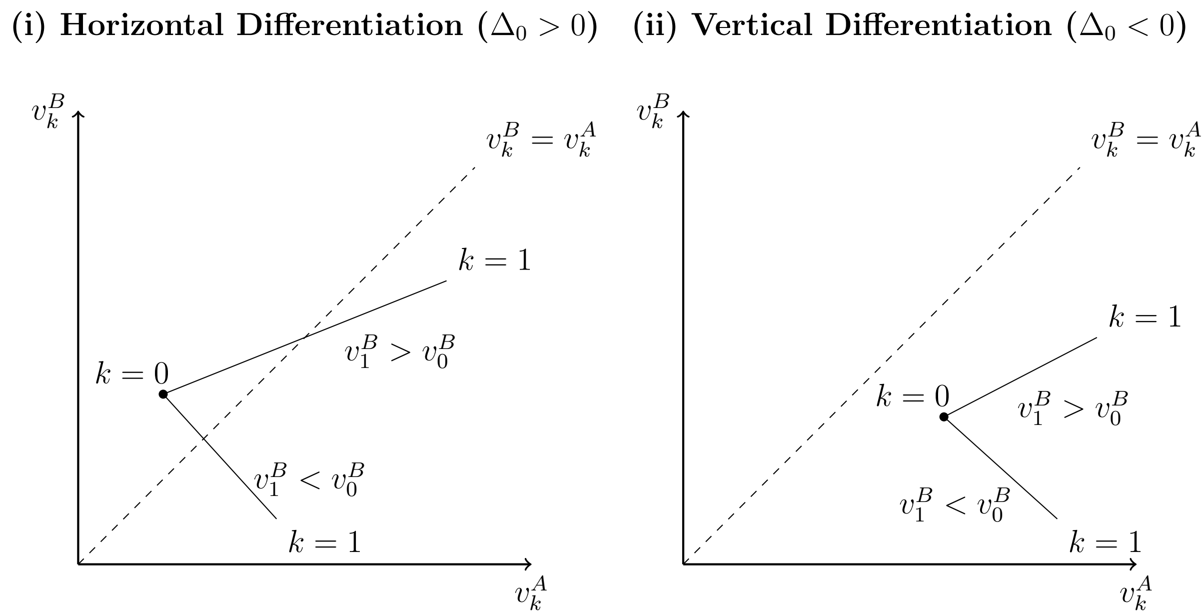

Figure 1: Illustration of Assumption 1. In each panel, the upward-sloping line captures positively correlated valuation for the two products, whereas the downward-sloping line captures negatively correlated valuations.

The assumptions made so far leave substantial flexibility regarding the nature of competitive interaction. They are compatible with the following cases.

Definition 1. *(i) The market is described by **horizontal differentiation** if $v_0^A < v_0^B$ and by **vertical differentiation** if $v_0^A > v_0^B$.*
*(ii) Preferences for the two products are **positively correlated** if $v_1^B > v_0^B$, **negatively correlated** if $v_1^B < v_0^B$.*

Figure 1 illustrates these cases. With horizontal differentiation (Figure 1(i)), consumers with sufficiently small k prefer product B to A at equal prices, whereas with vertical differentiation all consumers prefer A.[4] Under horizontal differentiation, Assumption 2 is essentially just a notational convention. To see this, note that Assumptions 2(i) and 2(iii) can be made to hold for arbitrary labeling of products as A and B by suitably indexing consumers.[5] Then one labeling of the two products will also satisfy Assumption 2(ii). Thus, under horizontal differentiation, Assumption 2 always holds. Under vertical differentiation, Assumption 2 is more restrictive. For Assumption 2(iii) to hold, A must be the better product. Consumers can always be ordered so that Assumption 2(i) holds. However, Assumption 2(ii), the single-crossing condition, has bite: It requires that the support line is flatter than the 45-degree line, as depicted in Figure 1(b). This is an actual restriction, implying that, as k increases, the additional WTP increase for the better product dominates the increase in WTP for the competing product.

The remaining assumptions jointly assure the existence of a full coverage equilibrium where all consumers are served and both firms sell a positive output.

Assumption 3. *(i)* $\Delta_1 - 2\Delta_0 + \Delta_C > 0$

(ii) $2\Delta_1 - \Delta_0 - \Delta_C > 0$

(iii) (a) $v_0^A > \frac{2c_A + c_B - \Delta_0 + 2\Delta_1}{3}$ *and (b)* $\min\{v_0^B, v_1^B\} > \frac{\Delta_1 - 2\Delta_0 + c_A + 2c_B}{3}$

Together, parts (i) and (ii) will be shown to guarantee that, in the proposed equilibrium, the indifferent consumer k^* is located in $(0, 1)$, the interior of the type space. Part (iii) is a sufficient conditions making sure that the market remains covered if at most one firm deviates from the equilibrium price. Specifically, (iii)(a) can be shown to guarantee that all consumers (including those who buy B in equilibrium) would obtain a positive net valuation from consuming good A at the equilibrium price. (iii)(b) is the corresponding statement for product B. Condition (iii) obviously holds as long as all valuations are high enough and costs sufficiently low.

The following result shows that it is not only possible to simultaneously satisfy all assumptions, but that all four constellations depicted in Figure 1 arise for suitable parameter vectors satisfying the assumptions. To see this, it suffices to focus on the case of zero marginal costs.

Proposition 1. *Suppose* $c_A = c_B = 0$. *Each combination of differentiation types (horizontal or vertical) and WTP correlations (positive or negative) is satisfied for a non-degenerate set of parameter vectors* $(v_0^A, v_1^A, v_0^B, v_1^B) \in \mathcal{R}_+^4$ *for which Assumption 2 and 3 hold.*

Figure 2 illustrates this result. The two textbook cases described in the introduction and the alternatives of horizontal differentiation combined with positive correlation and vertical differentiation combined with negative correlation all arise.

[4] W.l.o.g., Assumption 2(iii) rules out the possibility that all consumers prefer B at equal prices.

[5] For each product, by horizontal differentiation, one of the two most extreme consumers prefers it to the other one. Thus, if we denote the extreme consumer who prefers product A as $k = 1$, Assumptions 2(i) and 2(iii) both hold. The argument for product B is similar.

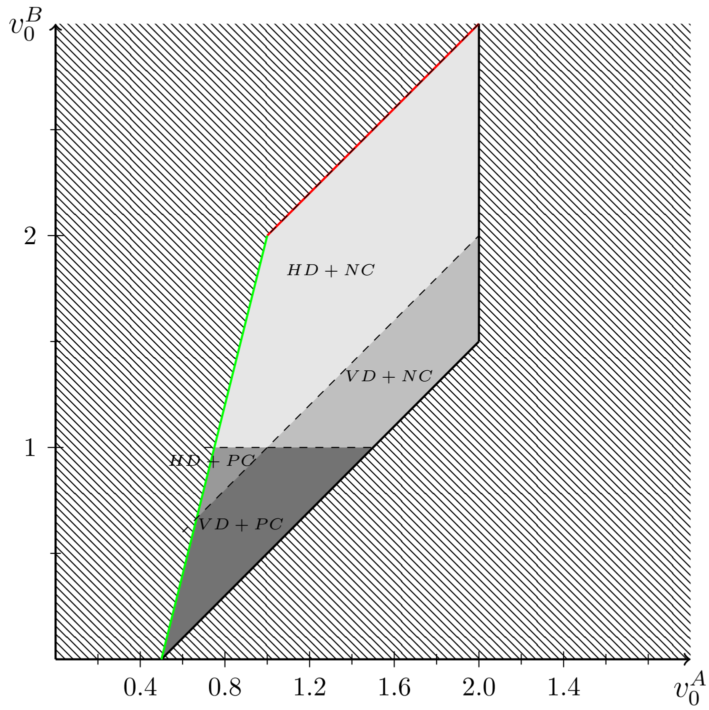

Figure 2: Illustration of Proposition 1. The figure depicts the values of v_0^A and v_0^B for which each combination of differentiation and correlation exists. HD (VD) stand for horizontal (vertical) differentiation, PC (NC) for positive (negative) correlation. The figure is drawn for $v_1^A = 2$ and $v_1^B = 1$.

3 Examples

The primitives of the model are the two cost parameters and the valuation distribution—nothing else matters for the equilibrium of the price game. It is nonetheless instructive to see that there are multiple ways in which the properties of the valuation distribution can be derived from underlying properties of the economic environment.

3.1 Textbook Models

As mentioned above, two simple models of differentiated duopoly fit the framework.

The Hotelling Model with Linear Transportation Cost In line with the standard Hotelling setting, interpret the type k of a consumer on the unit interval as his location. Firms A and B are located at $k = 1$ and $k = 0$, respectively. With linear transportation cost $t > 0$ and gross valuations $v > 0$, the valuations are therefore $v_k^A = v - t(1-k)$ and $v_k^B = v - tk$. Therefore, the valuation vectors are $(v_0^A, v_0^B) = (v - t, v)$ for consumer $k = 0$ and $(v_1^A, v_1^B) = (v, v-t)$ for $k = 1$. Thus, the support slopes downward from $(v-t, v)$ to $(v, v-t)$, and it is symmetric at the diagonal. Assumptions 1 and 2 always hold. If the firms have identical and constant marginal costs, Assumption 3 merely requires that v is large enough. More generally, the framework can cater for cost and demand asymmetries as long as they are sufficiently small.[6]

[6]With quadratic transportation costs, the support would be convex to the origin, thus violating Assumption 1.

The Shaked-Sutton Model of Vertical Differentiation In line with standard models of vertical differentiation, assume that products have qualities $s_A > s_B > 0$.[7] Further suppose that consumers have types $\theta \in [\underline{\theta}, \overline{\theta}]$, where $\underline{\theta} > 0$ and $\overline{\theta} := \underline{\theta} + 1$, and the type distribution is uniform. Moreover, suppose that type θ values product i at θs_i. The model fits into the framework of Section 2 with $v_k^i = (\underline{\theta} + k)s_i$, and Assumptions 1 and 2 hold without further restrictions. Clearly, the model satisfies the conditions for vertical differentiation ($v_0^A > v_0^B$) and positive correlation ($v_1^B > v_0^B$). Graphically, in Figure 1, the support is upward-sloping and does not intersect the diagonal. Finally, focusing for simplicity on the case of symmetric costs ($c_A = c_B$), Assumption 3 holds if one assumes in addition that $\underline{\theta} < 1$ and $c + \frac{(2\underline{\theta}+1)(s^A - s^B)}{3} < \underline{\theta} s^A$.[8]

In these models, horizontal differentiation coincides with negative correlation and vertical differentiation with positive correlation, so that only two of the four cases depicted in Figure 1 arise. In the following, I will discuss an alternative setting where the type of differentiation is independent of the type of correlation.

3.2 Objective and Subjective Quality Differentiation

Many products can be regarded as multi-dimensional objects, where the overall valuation of a consumer for the product is a function of his assessment of the valuation for each individual characteristic. Some dimensions, such as durability, have strong objective aspects, others, such as style, are intrinsically subjective. This is a natural setting in which all four combinations of differentiation type and WTP correlation can arise.

To illustrate the point, I assume that the value of each product $i \in \{A, B\}$ can be decomposed into a *quality* component $\omega^i \in \mathcal{R}^+$ that is fully objective and a *style* component $\sigma_k^i \in \mathcal{R}$ that is subjective so that

$$v_k^i = \omega^i + \sigma_k^i. \tag{2}$$

I further assume that, for all $k \in [0, 1]$,

$$\sigma_k^i = \sigma_0^i + k\left(\sigma_1^i - \sigma_0^i\right).$$

Thus, clearly, Assumption 1 holds. Assumption 2 requires

$$\begin{aligned} \sigma_1^A &> \sigma_0^A; \\ \sigma_1^A - \sigma_0^A &> \sigma_1^B - \sigma_0^B; \\ \omega^A + \sigma_1^A &> \omega^B + \sigma_1^B. \end{aligned}$$

These conditions all hold if σ_1^A is large enough. Horizontal differentiation applies whenever

$$\omega^A + \sigma_0^A < \omega^B + \sigma_0^B, \tag{3}$$

while vertical differentiation corresponds to

$$\omega^A + \sigma_0^A > \omega^B + \sigma_0^B. \tag{4}$$

[7]The following treatment adapts Section 7.5 in Tirole (1989) to the setting of Section 2.

[8]These two conditions correspond to parts (i) and (iii) of Assumption 2, and they immediately imply Assumptions (ii) and (iv), respectively. The first condition is equivalent to Assumption 1 in Tirole (1989); the latter condition implies Assumption 2 in Tirole (1989).

Finally, positive (negative) correlation applies if $\sigma_1^B > (<) \sigma_0^B$.

By applying Proposition 1, I show in Section A.3 in the appendix that, for suitable parameter vectors, all four cases in Figure 1 can arise, rather than only the two that were discussed in the previous subsection. For instance, horizontal differentiation is consistent with positive correlation if product A has a better objective component, but consumers with low k place much higher subjective value on B than on A.[9] Similarly, vertical differentiation can go hand in hand with negative correlation if product A has a sufficiently strong objective advantage over product B, but there is disagreement about the subjective component so that consumers with high valuation σ_k^A at the same time having low valuation σ_k^B for product B. Importantly, however, even for type $k = 1$, the agreement about the objective component captured by $\omega^A - \omega^B$ must dominate the disagreement about the subjective component captured by $\sigma_0^B - \sigma_0^A$ so that (4) holds.

4 Analysis of the Price Game

I now analyze the price game, returning to the general setting of Section 2. Section 4.1 characterizes the price equilibrium. Section 4.2 deals with the determinants of equilibrium outputs and profits.

4.1 Equilibrium Characterization

I now show that, under suitable conditions, the price game has an equilibrium with full coverage, where consumers with k below a cut-off k^* buy product B and those with $k > k^*$ buy product A. For prices (p_A, p_B), consumer k is indifferent between the products if

$$v_0^B + k\left(v_1^B - v_0^B\right) - p_B = v_0^A + k\left(v_1^A - v_0^A\right) - p_A. \tag{5}$$

This condition for the cut-off is equivalent with

$$k = \frac{\Delta_p - \Delta_0}{\Delta_1 - \Delta_0}. \tag{6}$$

This value lies strictly between 0 and 1 if and only if

$$\Delta_0 < \Delta_p < \Delta_1. \tag{7}$$

Using the uniformity of the consumer distribution and assuming full coverage, (6) immediately gives demand functions for prices satisfying (7) as

$$x_A(p_A, p_B) = \frac{\Delta_1 - \Delta_p}{\Delta_1 - \Delta_0} \tag{8}$$

$$x_B(p_B, p_A) = \frac{\Delta_p - \Delta_0}{\Delta_1 - \Delta_0} \tag{9}$$

The following result identifies the unique second-stage equilibrium.

[9] From condition (3), the requirement is $0 < \omega^A - \omega^B < \sigma_0^B - \sigma_0^A$.

Proposition 2. *Suppose that Assumptions 1-3 hold. Then the game has an interior equilibrium with full coverage. Equilibrium prices are*

$$p_A^* = \frac{c_B + 2c_A - \Delta_0 + 2\Delta_1}{3}$$
$$p_B^* = \frac{c_A + 2c_B + \Delta_1 - 2\Delta_0}{3}$$

Importantly, this equilibrium applies no matter whether differentiation is horizontal or vertical and whether correlation is positive or negative. These properties of preferences and products are only relevant for the equilibrium to the extent that they affect the WTP differences Δ_1 and Δ_0. Both prices are increasing in Δ_1, but decreasing in Δ_0. The difference reflects the intuition that an increase in Δ_1 (the WTP of the consumer with the strongest preferences for product A) increases the heterogeneity in consumer valuations for the two products, thereby softening competition, whereas an increase in Δ_0 reduces heterogeneity, thereby intensifying competition. Finally, both prices are increasing in costs, with the price effects of own cost increases dominating those of competitor cost increases. Proposition 2 obviously implies the following result on equilibrium margins $m_i^* := p_i^* - c_i$.

Corollary 1. ***Determinants of Margins***

(i) A reduction in Δ_C or a uniform increase of Δ_0 and Δ_1 increases the margin of A and reduces the margin of B.

(ii) An increase in Δ_1 increases the margins of both firms.

(iii) An increase in Δ_0 reduces the margins of both firms.

To see the intuition for the first result in (i), note that a lower Δ_C can result from a reduction in c_A or an increase in c_B. For instance, a reduction in c_A has a direct positive effect on the margin of firm A, which is only partly compensated by the resulting reduction of the equilibrium price p_A^*. For the rival B, the cost reduction of A only reduces the equilibrium price p_B^*, without any positive effect from lower costs, so that its margin must fall. (ii) reflects a competition-softening effect arising because, as Δ_1 increases (and Δ_0 stays fixed), the consumers with strong preferences for firm A now have even higher WTP for that firm so that consumer heterogeneity increases. If Δ_0 increases (for fixed Δ_1), the opposite effect occurs because consumers become more similar: As competition becomes more intense, margins decrease (iii).

In contrast with the effects of WTP increases of firm 1 that are biased towards consumer types with particularly high or low valuations, the second result in (i) states that a uniform increase of Δ_0 and Δ_1 and thus of all Δ_k by the same amount has the same qualitative effect as a cost reduction of firm 1. Such an unbiased WTP increase combines the opposing effects (ii) and (iii). For firm A, the positive effect on margins from higher Δ_1 dominates the adverse effect of lower Δ_0. For firm B, the adverse competition-intensifying effect of increasing Δ_0 dominates the positive competition-softening effect of increasing Δ_1.

4.2 Determinants of Outputs and Profits

I now describe how equilibrium outputs and profits depend on primitives.

4.2.1 Determinants of Outputs

Equilibrium outputs result from inserting equilibrium prices into demand functions:

$$x_A^* = x_A^*(\Delta_0, \Delta_1, \Delta_C) = \frac{2\Delta_1 - \Delta_0 - \Delta_C}{3\,(\Delta_1 - \Delta_0)} \tag{10}$$

$$x_B^* = x_B^*(\Delta_0, \Delta_1, \Delta_C) = \frac{\Delta_1 - 2\Delta_0 + \Delta_C}{3\,(\Delta_1 - \Delta_0)} \tag{11}$$

It is natural to conjecture that a firm's market share increases whenever it becomes stronger, meaning that Δ_0 or Δ_1 increases or Δ_C falls. I will now substantiate this result while clarifying its limitations. To this end, it will be useful to distinguish between different parameter regimes, as in Figure 3.

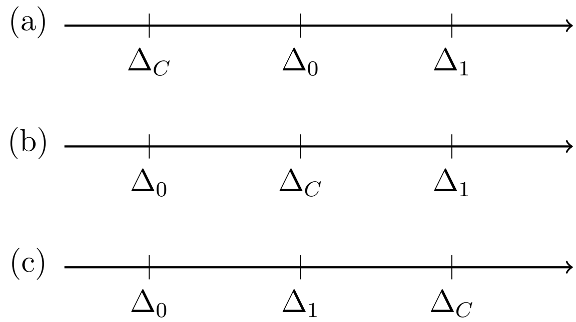

Figure 3: Parameter Constellations. In case (a), increases in Δ_0 increase the market share of firm A; while increases in Δ_1 reduce it; in case (b), increases in Δ_0 and Δ_1 both increase the market share of firm A; in case (c), increases in Δ_0 reduce the market share of firm A; whereas increases in Δ_1 increase it.

To understand the three constellations, first assume that $\Delta_C = 0$. Then only cases (a) and (b) can emerge, with the former corresponding to vertical and the latter to horizontal differentiation. If $\Delta_C > 0$, then case (c) is also possible. The distinction between constellations (a)-(c) is essential for the comparative statics of market shares with respect to WTP parameters.

Corollary 2. ***Determinants of Market Shares***

(i) A reduction in Δ_C or a uniform increase in Δ_0 and Δ_1 increases the market share of firm A.

(ii) If $\Delta_1 > \Delta_C$, then an increase in Δ_0 increases the market share of firm A. If $\Delta_1 < \Delta_C$, an increase in Δ_0 reduces the market share of firm A.

(iii) If $\Delta_0 < \Delta_C$, an increase in Δ_1 increases the market share of firm A. If $\Delta_0 > \Delta_C$, an increase in Δ_1 reduces the market share of firm A.

Result (i) confirms the conjecture that, if a firm gets stronger in the sense of having lower costs or higher WTP for all consumers, then its market share increases: In the former case, firm A's costs fall in relative terms, it reduces the price. The competitor follows suit, but by a smaller amount. Therefore, its reaction will not fully compensate the direct market share effect of firm A's price reduction. In the latter case, the higher WTP for firm A has a direct positive effect on its market share, which is reinforced by the price effects. By contrast, results (ii) and (iii) show that the effects of biased changes in Δ_0 or Δ_1 on market shares depend on which of the three parameter constellations in Figure 3 arises. In the case without cost asymmetries ($\Delta_C = 0$), constellation (a) in Figure 3 corresponds to vertical differentiation. Corollary 2 states that, in this case, an increase in Δ_0 increases the market share of firm A, whereas an increase in Δ_1 decreases it. As constellation (b) in Figure 3 corresponds to horizontal differentiation for $\Delta_C = 0$, Corollary 2 says that, in this case, still assuming cost symmetry, increases in Δ_0 and Δ_1 both increase the market share of firm A. Finally, constellation (c) in Figure 3 can only arise with sufficiently large cost disadvantages of firm A. In this case, an increase in Δ_1 increases the market share of firm A, whereas an increase in Δ_0 decreases it.

The intuition for (ii) and (iii) relies on the interplay of direct demand effects and price-mediated effects. For instance, for fixed prices, an increase in Δ_1 shifts demand to firm A. However, this direct effect is dampened by the price reactions. As the increased differentiation in consumer tastes softens competition, both prices increase, but more so for firm A that exploits the increased willingness to pay for its product. With $\Delta_C = 0$, the case distinction in (iii) amounts to the difference between horizontal and vertical differentiation. In the former case ($\Delta_0 < 0$), the direct demand effect dominates, resulting in an output shift to firm A. More surprisingly, in the latter case ($\Delta_0 > 0$), the price-induced effect dominates the direct effect, and the market share of firm B increases. Thus, an improvement in the relative WTP of its most loyal consumers for its product need not increase A's market share – with vertical differentiation, A may exploit the increasing WTP for its product instead.

An increase in Δ_0 (reflecting an increase in the WTP of firm B's most loyal consumers for firm A's product) similarly affects market shares directly and via price adjustments. However, as discussed above, the demand shift intensifies competition, inducing stronger price reductions for than for firm A. As $\Delta_1 > 0$ by Assumption 2(iii), Part (ii) of Corollary 2 implies that, for symmetric costs, the direct effect now always dominates, so that the market share of firm A increases nonetheless. The negative effect of an increase in Δ_0 on the market share of A when it has a high cost disadvantage reflects the intuition that a relative weak firm loses market share when it is no longer protected from competition.

Finally, the observation that, in constellation (c) in in Figure 3, the market share of firm A increases only after an increase in Δ_1, but falls after an increase in Δ_0 reflects the intuition that firm A is a weak firm because of its cost disadvantage and therefore benefits if it is shielded from competition by a higher Δ_1 or lower Δ_0.

4.2.2 The Determinants of Profits

The expressions for equilibrium prices (Proposition 2) and outputs ((13) and (12)) directly yield equilibrium profits as

$$\pi_A^* = \pi_G^*(\Delta_0, \Delta_1, \Delta_C) = \frac{(2\Delta_1 - \Delta_0 - \Delta_C)^2}{9\,(\Delta_1 - \Delta_0)} \qquad (12)$$

$$\pi_B^* = \pi_B^*(\Delta_0, \Delta_1, \Delta_C) = \frac{(\Delta_1 + \Delta_C - 2\Delta_0)^2}{9(\Delta_1 - \Delta_0)} \tag{13}$$

In the following, I investigate in more detail how profits depend on primitives.

Corollary 3. ***Determinants of Profits***

(i) A reduction in Δ_C, the relative costs of firm A, or a uniform increase in the WTP differential Δ_k increases the profits of firm A and reduces those of B.

(ii) An increase in Δ_1 increases the profits of firm A. It increases those of firm B if and only if $\Delta_1 > \Delta_C$.

(iii) An increase in Δ_0 reduces the profits of firm B. It increases those of firm A if and only if $\Delta_0 > \Delta_C$.

These results reflect the interplay of the margin and output effects (Corollaries 1 and 2). The first part of (i) unsurprisingly states that a firm's profit increases if the cost differential changes in its favor (as its equilibrium outputs and margins increase). The intuition behind the result for uniform changes in WTP is similar. (ii) states that an increase of Δ_1 increases firm A's profits and, under a mild additional condition that always holds with symmetric costs by Assumption 2, the profits of firm B as well. This again reflects the competition-softening effect of the parameter change. (iii) states that firm B's profits fall after an increase of Δ_0, whereas a further condition is needed for an increase in the profits of firm A. With symmetric costs, this condition holds for vertical differentiation, but not for horizontal differentiation. Intuitively, both firms suffer from increasing competition following the increase in Δ_0, whereas only firm A benefits from the improvement in its relative position.

None of the above depends on the sign of the correlation between the valuations for the two products. By contrast, whether differentiation is horizontal or vertical is central for the effects of biased WTP changes on market shares and profits.

4.2.3 Comparison

Table 1 summarizes the output and profit effects for the case of symmetric costs.

	Green Market Share	Green Profit	Brown Profit
Horizontal Differentiation ($\Delta_0 < 0 < \Delta_1$)	$+$ $(+)$	$+$ $(-)$	$+$ $(-)$
Vertical Differentiation ($0 < \Delta_0 < \Delta_1$)	$-$ $(+)$	$+$ $(+)$	$+$ $(-)$

Table 1: Market Share and Profit Effects of Increasing $\Delta_1(\Delta_0)$ with Symmetric Costs

The table highlights how the effects of heterogeneous WTP increases on outputs and profits differ for horizontal and vertical differentiation. In the former case, biased improvements always increase market shares but not necessarily profits; in the latter case, biased improvements always increase profits but not necessarily outputs.

5 Conclusions

This short paper provides a simple framework, in which standard models of horizontal and vertical differentiation can be analyzed in a unified way. The framework uses the willingness-to-pay distribution of consumers as the primitive of the model, and it expresses the equilibrium as a function of these distributions. The approach allows us to separate the role of the type of differentiation (horizontal or vertical) from the type of correlation (positive or negative). It turns out that the former distinction plays a critical role for some comparative statics results, whereas the latter does not.

This framework has been chosen in the simplest possible way to incorporate standard textbook models of differentiation. Obvious candidates for generalization include (i) the consumer distribution (for fixed support), (ii) the support (which could a non-linear curve or a two-dimensional subset of $\mathcal{R}_+^2$) and the number of firms.

A Appendix

This appendix contains proofs and calculations required to corroborate the results mentioned in the main text.

A.1 Proof of Proposition 1

It suffices to show that the desired conditions can be fulfilled even in the special case that $c_A = c_B = 0$, in which the conditions of Assumption 2(i)-(iii) and 3(i)-(iii) can be rewritten as follows.

$$v_1^A > v_0^A \tag{14}$$
$$v_1^A > v_0^A + v_1^B - v_0^B \tag{15}$$
$$v_1^A > v_1^B \tag{16}$$
$$v_1^A > v_1^B + 2v_0^A - 2v_0^B \tag{17}$$
$$v_1^A > v_1^B + \frac{v_0^A - v_0^B}{2} \tag{18}$$
$$v_1^B + 2v_0^A - \frac{v_0^B}{2} > v_1^A \tag{19}$$
$$4v_1^B + 2v_0^A - 2v_0^B > v_1^A \tag{20}$$
$$v_1^B + 2v_0^A + v_0^B > v_1^A \tag{21}$$

In the following, I will show that, for each combination of differentiation and correlation, all conditions (14)-(21) can be satisfied simultaneously. As (19) implies (21), the latter condition can be ignored below.

Horizontal Differentiation and Negative Correlation: It is straightforward to show that, if $v_0^A < v_0^B$, the requirement for horizontal differentiation, and $v_0^B > v_1^B$, the requirement for negative correlation, both hold, then all conditions for Assumption 2 and 3 (14)-(21) hold whenever (14), (16), (19) and (20) do. These four conditions as well as $v_0^A < v_0^B$ and $v_0^B > v_1^B$ obviously hold strictly if $v_0^B = v_1^A =: v$ and $v_1^B = v_0^A = \lambda v$ for some $\lambda \in (\frac{1}{2}, 1)$. Small perturbations of the parameter vector do not change this.

Horizontal Differentiation and Positive Correlation: It is straightforward to show that if the assumptions for horizontal differentiation ($v_0^A < v_0^B$) and positive correlation ($v_0^B < v_1^B$) hold, then equations (14)-(21) all hold if conditions (16) and (19) both hold as well. To see that it is possible to fulfill (16),(19), $v_0^A < v_0^B$ and $v_0^B < v_1^B$ simultaneously, first choose $v_0^A < v_0^B < v_1^B$, but such that $4v_0^A > v_0^B$. Then $v_1^B + 2v_0^A - \frac{v_0^B}{2} > v_1^B$, so that by appropriate choice of v_1^A, conditions (16) and (19) can both be made to hold.

Vertical Differentiation and Positive Correlation: It is straightforward to show that the assumptions for vertical differentiation ($v_0^A > v_0^B$) and positive correlation ($v_0^B < v_1^B$) and the conditions for Assumption 2 and 3 (equations (14)-(21)) all hold if

$$v_1^B > v_0^B \tag{22}$$
$$v_1^A > v_0^A > v_0^B \tag{23}$$
$$v_1^A > v_1^B + 2v_0^A - 2v_0^B \tag{24}$$
$$v_1^B + 2v_0^A - \frac{v_0^B}{2} > v_1^A \tag{25}$$

Conditions (22) and $v_0^A > v_0^B$ imply that the right-hand side of (24) is bounded below by v_0^A. Thus, $v_1^A > v_0^A$ can be ignored in (23). To jointly fulfill the remaining inequalities, first choose v_0^A, v_0^B and v_1^B so that (22) holds and $v_0^A > v_0^B$. Then (24) and (25) hold if and only if

$$v_A^1 \in \left(v_1^B + 2v_0^A - 2v_0^B, v_1^B + 2v_0^A - \frac{v_0^B}{2}\right), \tag{26}$$

which is a non-empty interval.

Vertical Differentiation and Negative Correlation: It is straightforward to show that the conditions for Assumption 2 and 3 (equations (14)-(21)) as well as those for vertical differentiation ($v_0^A > v_0^B$) and negative correlation ($v_0^B > v_1^B$) hold if and only if

$$v_1^A > v_0^A > v_0^B > v_1^B \tag{27}$$
$$v_1^A > v_1^B + 2v_0^A - 2v_0^B \tag{28}$$
$$v_1^B + 2v_0^A - \frac{v_0^B}{2} > v_1^A \tag{29}$$

Choose $v_0^A > v_0^B > v_1^B$ so that $v_0^A - v_1^B$ is sufficiently small. As

$$v_1^B + 2v_0^A - 2v_0^B < 2v_0^A + v_1^B - \frac{v_0^B}{2},$$

the three requirements on v_1^A can be satisfied by choosing v_1^A slightly larger than v_0^A.

A.2 Derivations for Figure 2

The figure is depicted for $v_1^A = 2$ and $v_1^B = 1$. Inserting these values into equations (14)-(21), one can show that (14)-(18) all hold provided the following four conditions do:

$$2 > v_0^A \tag{30}$$
$$v_0^B > v_0^A - \frac{1}{2} \tag{31}$$
$$v_0^B < v_0^A + 1 \tag{32}$$
$$4v_0^A > 2 + 2v_0^B. \tag{33}$$

For a parameter vector given by v_0^A, v_0^B and $v_1^A = 2$, $v_1^B = 1$, this corresponds to the parameter area described by the rectangle in Figure 2. The four regimes within the reflect reflect the signs of $v_0^A - v_0^B$ and $v_1^B - v_0^B$, respectively. They thus correspond to the four different combinations of differentiation type and correlation.

A.3 Derivations for Section 3

Objective and Subjective Quality Differentiation: To show that, in this model, all four combinations of differentiation and correlation type can be generated, it suffices to show that every valuation vector $(v_0^A, v_1^A, v_0^B, v_1^B)$ can be constructed by suitable choices of (ω_A, ω_B) and $(\sigma_0^A, \sigma_1^A, \sigma_0^B, \sigma_1^B)$, because then Proposition 1 immediately implies the result. This corresponds to choosing the six variables (ω_A, ω_B) and $(\sigma_0^A, \sigma_1^A, \sigma_0^B, \sigma_1^B)$ so as to solve the four equations (2) for any fixed valuation vector $(v_0^A, v_1^A, v_0^B, v_1^B)$ satisfying Assumptions 2 and 3, which is trivial.

A.4 Proof of Proposition 2

Proof. Suppose prices are such that both firms have positive demand and that the market is covered. Then, using (6), Firm B's profits are $(p_B - c_B)\frac{(p_A-p_B)-\Delta_0}{\Delta_1-\Delta_0}$. Solving the FOC gives the reaction function

$$p_B = \frac{p_A + c_B - \Delta_0}{2} \tag{34}$$

Firm A's profits are given as $(p_A - p_A)\left(1 - \frac{(p_A-p_B)-\Delta_0}{\Delta_1-\Delta_0}\right)$, yielding

$$p_A = \frac{p_B + p_A + \Delta_1}{2} \tag{35}$$

Solving the system of equations given by (34) and (35) yields p_B^* and p_A^* as stated in Proposition 2 as a candidate equilibrium.

I now show that Assumptions 1 and 2 guarantee that the FOCs define an equilibrium. The second-order conditions obviously hold for both profit functions. Assumptions 3(i) and (ii) make sure that both firms have positive market shares and margins in the proposed equilibrium and thus earn positive profits. Assumption 3(iii) guarantees that all consumers obtain a positive net surplus from buying either of the two products at the equilibrium price. Thus, no matter how a firm deviates, each consumer buys from one of the two firms, and the deviating firm's demand will be determined by the indifferent consumer (as in (6)) unless it deviates so much that demand becomes 0 or 1. First- and second-order conditions jointly guarantee that deviations where both firms share the market are not profitable. They also imply that lowering prices so strongly that demand becomes 1 cannot be profitable. A fortiori, lowering prices even further cannot be profitable. Finally, increasing prices so much that demand becomes zero is obviously not profitable. □

A.5 Comparative Statics

This section provides details for the comparative statics results in Section 4. I first sketch the arguments for Corollaries 1 and 2. Then I deal with the slightly harder results.

A.5.1 Remarks on Corollaries 1 and 2

Corollary 1 is obvious from Proposition 2, using the fact that the equilibrium margins are given as $p_i^* - c_i$ for $i \in \{A, B\}$. Similarly, Corollary 2(i) is an immediate implication of the formulas for equilibrium outputs ((10) and (11)). (ii) holds because

$$\frac{\partial}{\partial \Delta_0}\frac{2\Delta_1 - \Delta_0 - \Delta_C}{3(\Delta_1 - \Delta_0)} = \frac{\Delta_1 - \Delta_C}{3(\Delta_0 - \Delta_1)^2} > 0 \text{ if and only if } \Delta_1 > \Delta_C$$

(iii) holds because

$$\frac{\partial}{\partial \Delta_1} \frac{2\Delta_1 - \Delta_0 - \Delta_C}{3(\Delta_1 - \Delta_0)} = -\frac{\Delta_0 - \Delta_C}{3(\Delta_0 - \Delta_1)^2} > 0 \text{ if and only if } \Delta_0 < \Delta_C.$$

A.5.2 Proof of Corollary 3

Recall that equilibrium profits are

$$\pi_A = \frac{1}{9} \frac{(2\Delta_1 - \Delta_0 - \Delta_C)^2}{(\Delta_1 - \Delta_0)}$$

$$\pi_B = \frac{1}{9} \frac{(\Delta_1 + \Delta_C - 2\Delta_0)^2}{\Delta_1 - \Delta_0}$$

(i) Using Assumption 3(ii), this follows directly from

$$-\frac{\partial}{\partial \Delta_C} \frac{1}{9} \frac{(2\Delta_1 - \Delta_0 - \Delta_C)^2}{(\Delta_1 - \Delta_0)} = \frac{2\Delta_0 - 4\Delta_1 + 2\Delta_C}{9(\Delta_0 - \Delta_1)} > 0$$

and

$$\frac{\partial}{\partial \Delta_C} \frac{1}{9} \frac{(\Delta_1 + \Delta_C - 2\Delta_0)^2}{\Delta_1 - \Delta_0} = \frac{2\Delta_0 - 4\Delta_1 + 2\Delta_C}{9(\Delta_0 - \Delta_1)} > 0.$$

(ii) First note that

$$\frac{\partial}{\partial \Delta_1} \frac{1}{9} \frac{(2\Delta_1 - \Delta_0 - \Delta_C)^2}{(\Delta_1 - \Delta_0)} = \frac{1}{9} (\Delta_0 - 2\Delta_1 + \Delta_C) \frac{3\Delta_0 - 2\Delta_1 - \Delta_C}{(\Delta_0 - \Delta_1)^2}. \tag{36}$$

Next, Assumption 3(i) implies that

$$3\Delta_0 - 2\Delta_1 - \Delta_C < 3\Delta_0 - 2(2\Delta_0 - \Delta_C) - \Delta_C = \Delta_C - \Delta_0.$$

Similarly, Assumption 3(ii) implies that

$$3\Delta_0 - 2\Delta_1 - \Delta_C < 3\Delta_0 - 2\left(\frac{\Delta_0 + \Delta_C}{2}\right) - \Delta_C = 2\Delta_0 - 2\Delta_C.$$

These two conditions can only hold simultaneously if and only if

$$3\Delta_0 - 2\Delta_1 - \Delta_C < 0. \tag{37}$$

Using Assumption 3(ii) and (37), (36) is positive, so that an increase in Δ_1 increases the profits of firm A.
Using Assumption 3(i), the result for firm B follows from

$$\frac{\partial}{\partial \Delta_1} \frac{1}{9} \frac{(\Delta_1 + \Delta_C - 2\Delta_0)^2}{\Delta_1 - \Delta_0} = \frac{1}{9} (\Delta_1 - \Delta_C) \frac{\Delta_1 + \Delta_C - 2\Delta_0}{(\Delta_0 - \Delta_1)^2} > 0.$$

(iii) First,

$$\frac{\partial}{\partial \Delta_0} \frac{1}{9} \frac{(\Delta_1 + \Delta_C - 2\Delta_0)^2}{\Delta_1 - \Delta_0} = \frac{1}{9} (2\Delta_0 - 3\Delta_1 + \Delta_C) \frac{\Delta_1 + \Delta_C - 2\Delta_0}{(\Delta_0 - \Delta_1)^2}. \tag{38}$$

Next note that Assumptions 3(i) implies

$$2\Delta_0 - 3\Delta_1 + \Delta_C < 2\Delta_0 - 3\left(2\Delta_0 - \Delta_C\right) + \Delta_C = 4\Delta_C - 4\Delta_0$$

and Assumptions 3(ii) implies

$$2\Delta_0 - 3\Delta_1 + \Delta_C < 2\Delta_0 - 3\left(\frac{\Delta_0 + \Delta_C}{2}\right) + \Delta_C = \frac{1}{2}\left(\Delta_0 - \Delta_C\right).$$

Together, the two last inequalities thus imply that $2\Delta_0 - 3\Delta_1 + \Delta_C < 0$, so that the effect of an increase in Δ_0 on the profits of firm B is negative according to (38) and Assumption 3(i). Finally, the result for firm A follows directly from Assumption 3(ii) and

$$\frac{\partial}{\partial \Delta_0} \frac{1}{9} \frac{\left(2\Delta_1 - \Delta_0 - \Delta_C\right)^2}{\left(\Delta_1 - \Delta_0\right)} = -\frac{1}{9}\left(\Delta_0 - 2\Delta_1 + \Delta_C\right) \frac{\Delta_0 - \Delta_C}{\left(\Delta_0 - \Delta_1\right)^2}. \tag{39}$$

References

Belleflamme, P. and M. Peitz (2010): *Industrial Organization: Markets and Strategies*, Cambridge University Press.

Hotelling, H. (1929): "Stability in Competition," *Economic Journal*, 39, 41–57.

Perloff, J. M. and S. C. Salop (1985): "Equilibrium with Product Differentiation," *Review of Economic Studies*, 52, 107–120.

Salop, S. C. (1979): "Monopolistic Competition with Outside Goods," *Bell Journal of Economics*, 10, 141–156.

Schmutzler, A. (2024): "Providing Incentives for the Green Transition," Working paper, University of Zurich.

Shaked, A. and J. Sutton (1982): "Relaxing price competition through product differentiation," *The Review of Economic Studies*, 49, 3–13.

Tirole, J. (1989): *The Theory of Industrial Organization*, MIT Press.